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Research Article

Milne's Estimates via Conformable Fractional Multiplicative Integrals

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Abstract

This paper explores Milne's inequality in the context of multiplicative conformable fractional integrals, a recent extension of classical fractional calculus rooted in non-Newtonian analysis. Leveraging recent developments in multiplicative calculus, we establish a new fundamental identity that underpins the derivation of Milne-type inequalities for two classes of functions: (i) those whose *-derivatives in *-absolute value are multiplicative convex, and (ii) those with bounded *-derivatives. A numerical example, accompanied by graphical illustrations, is included to demonstrate the validity and effectiveness of the theoretical findings. Further, we extend our analysis to functions satisfying specific derivative constraints and present some applications to special means. The paper concludes with a summary of the main contributions and a discussion of promising avenues for future research in multiplicative fractional analysis.

1. Introduction

Quadrature formulas play a fundamental role in numerical analysis by providing efficient approximations of definite integrals. These methods are essential when dealing with functions that do not admit elementary antiderivatives or when integrating data obtained from experimental or computational sources. Among the most well-known quadrature rules are the trapezoidal rule, Simpson's rule, and other Newton-Cotes formulas, which approximate integrals using weighted sums of function values at specific sampling points.

A key aspect in the study of quadrature formulas lies in estimating the approximation error, especially for functions with limited regularity. In this context, convexity has proven to be a powerful tool for deriving error bounds and analyzing the performance of various numerical integration schemes. Several results have been established using this concept; see [1,2] for foundational works on convexity-based inequalities in quadrature error estimation.

In particular, the Milne rule is closely related to Simpson's rule and is known for its efficiency in certain integration problems. Although both formulas are based on three evaluation points, the Milne rule includes a corrector term, leading to distinct accuracy characteristics and error behaviors compared to Simpson's rule. The Milne quadrature rule has attracted significant interest in recent years, especially for its approximation accuracy and distinctive error characteristics. For some pioneering works addressing Milne's inequality using various types of convexity, we refer readers to [3–5].

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1.1. Multiplicative Calculus

The development of classical calculus, attributed to the independent works of Newton and Leibniz, marked a turning point in mathematical analysis. Their formulation of differentiation and integration provided powerful tools for modeling dynamic systems, enabling precise descriptions of rates of change and the accumulation of quantities. In this traditional framework, derivatives are defined through additive differences, while integrals emerge as limits of summations, both deeply rooted in linear structures.

A significant departure from this classical perspective emerged in the late twentieth century with the introduction of an alternative system known as multiplicative calculus, first formalized by Grossman and Katz [6]. This innovative approach redefines the basic operations of calculus: instead of relying on addition and subtraction, it employs multiplication and division as its foundational elements. As a result, multiplicative calculus offers a new analytical lens, particularly effective when dealing with exponential behaviors or multiplicative processes.

It is important to note that this framework applies only to positive real-valued functions, which inherently restricts its scope compared to classical calculus. Nevertheless, multiplicative calculus has found meaningful applications across diverse fields such as economics, biology, signal processing, and optimization theory. Its development since the 1970s has opened new avenues for interpreting functional behavior, especially in contexts where exponential growth or decay plays a central role.

At the heart of multiplicative calculus lies a reformulation of the derivative and integral concepts. The multiplicative derivative, for instance, captures the ratio of function values at infinitesimally close points, rather than their difference. This shift in perspective leads to a fundamentally different interpretation of change and accumulation. We now turn to a more detailed presentation of this concept.

Definition 1.1 (see [7]). *The multiplicative derivative of a positive function $\mathscr{V}$, represented by $\mathscr{V}^*$, is expressed as:*

$$\frac{d^*\mathscr{V}}{\mathrm{d}\ell} = \mathscr{V}^*(\ell) = \lim_{\mathfrak{h}\to 0}\left(\frac{\mathscr{V}(\ell+\mathfrak{h})}{\mathscr{V}(\ell)}\right)^{\frac{1}{\mathfrak{h}}}.$$

Remark 1.2. *Any positive differentiable function $\mathscr{V}$ naturally demonstrates multiplicative differentiability. The connection between $\mathscr{V}^*$ and $\mathscr{V}'$ is characterized by the subsequent relation:*

$$\mathscr{V}^*(\ell) = \exp\left\{(\ln\mathscr{V}(\ell))'\right\} = \exp\left\{\frac{\mathscr{V}'(\ell)}{\mathscr{V}(\ell)}\right\}.$$

Proposition 1.3 (see [7]). *Consider two functions $\mathscr{V}$ and $\mathscr{W}$ that are *-differentiable (in the multiplicative sense), and let $\mathscr{U}$ be a differentiable function. Furthermore, let c represent an arbitrary positive constant. Under these conditions, the following functions are also *-differentiable: $c\mathscr{V}$, $\mathscr{V}\mathscr{W}$, $\mathscr{V}+\mathscr{W}$, $\mathscr{V}/\mathscr{W}$, and $\mathscr{V}^{\mathscr{U}}$. Furthermore, the following properties hold:*

$$\begin{aligned}
&(c\mathscr{V})^*(\ell) = \mathscr{V}^*(\ell),\\
&(\mathscr{V}+\mathscr{W})^*(\ell) = \mathscr{V}^*(\ell)^{\frac{\mathscr{V}(\ell)}{\mathscr{V}(\ell)+\mathscr{W}(\ell)}}\mathscr{W}^*(\ell)^{\frac{\mathscr{W}(\ell)}{\mathscr{V}(\ell)+\mathscr{W}(\ell)}},\\
&(\mathscr{V}\mathscr{W})^*(\ell) = \mathscr{V}^*(\ell)\mathscr{W}^*(\ell),\\
&\left(\frac{\mathscr{V}}{\mathscr{W}}\right)^*(\ell) = \frac{\mathscr{V}^*(\ell)}{\mathscr{W}^*(\ell)},\\
&\left(\mathscr{V}^{\mathscr{U}}\right)^*(\ell) = \mathscr{V}^*(\ell)^{\mathscr{U}(\ell)}\mathscr{V}(\ell)^{\mathscr{U}'(\ell)},\\
&(\mathscr{V}\circ\mathscr{U})^*(\ell) = \mathscr{V}^*(\mathscr{U}(\ell))^{\mathscr{U}'(\ell)}.
\end{aligned}$$

Definition 1.4 (see [7]). *The multiplicative integral of a positive function $\mathscr{V}$, is expressed as:*

$$\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\{\mathscr{V}(\ell)\}^{\mathrm{d}\ell} = \exp\left\{\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\ln(\mathscr{V}(\ell))\,\mathrm{d}\ell\right\}.$$

Proposition 1.5 (see [7]). *Consider positive and Riemann integrable functions $\mathscr{V}$ and $\mathscr{W}$, then $\mathscr{V}$ and $\mathscr{W}$ are *-integrable and the following properties hold true.*

$$\begin{aligned}
&\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\left((\mathscr{V}(\ell))^k\right)^{\mathrm{d}\ell} = \left(\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}(\mathscr{V}(\ell))^{\mathrm{d}\ell}\right)^k, \quad k\in\mathbb{R},\\
&\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}(\mathscr{V}(\ell))^{\mathrm{d}\ell} = \int_{\mathfrak{z}_1}^{\sigma}(\mathscr{V}(\ell))^{\mathrm{d}\ell}\int_{\sigma}^{\mathfrak{z}_2}(\mathscr{V}(\ell))^{\mathrm{d}\ell},\ \mathfrak{z}_1<\sigma<\mathfrak{z}_2,\\
&\int_{\mathfrak{z}_1}^{\mathfrak{z}_1}(\mathscr{V}(\ell))^{\mathrm{d}\ell} = 1,\ \text{and}\ \int_{\mathfrak{z}_1}^{\mathfrak{z}_2}(\mathscr{V}(\ell))^{\mathrm{d}\ell} = \left(\int_{\mathfrak{z}_2}^{\mathfrak{z}_1}(\mathscr{V}(\ell))^{\mathrm{d}\ell}\right)^{-1},\\
&\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}(\mathscr{V}(\ell)\mathscr{F}(\ell))^{\mathrm{d}\ell} = \int_{\mathfrak{z}_1}^{\mathfrak{z}_2}(\mathscr{V}(\ell))^{\mathrm{d}\ell}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}(\mathscr{F}(\ell))^{\mathrm{d}\ell},\\
&\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\left(\frac{\mathscr{V}(\ell)}{\mathscr{W}(\ell)}\right)^{\mathrm{d}\ell} = \frac{\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}(\mathscr{V}(\ell))^{\mathrm{d}\ell}}{\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}(\mathscr{W}(\ell))^{\mathrm{d}\ell}}.
\end{aligned}$$

Theorem 1.6 (see [7]). *Let $\mathscr{V}$ be a positive *-differentiable function defined on the interval $[\mathfrak{z}_1,\mathfrak{z}_2]$, and let $\mathscr{W}:[\mathfrak{z}_1,\mathfrak{z}_2]\to\mathbb{R}$ be a differentiable function. Then, the function $(\mathscr{V}^*)^{\mathscr{W}}$ is *-integrable, and we have that*

$$\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\left(\mathscr{V}^*(\ell)^{\mathscr{W}(\ell)}\right)^{\mathrm{d}\ell} = \frac{\mathscr{V}(\mathfrak{z}_2)^{\mathscr{W}(\mathfrak{z}_2)}}{\mathscr{V}(\mathfrak{z}_1)^{\mathscr{W}(\mathfrak{z}_1)}}\times\frac{1}{\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\left(\mathscr{V}(\ell)^{\mathscr{W}'(\ell)}\right)^{\mathrm{d}\ell}}.$$

Lemma 1.7 (see [7]). *Consider a positive $*$differentiable function $\mathscr{V}$ defined on $[\mathfrak{z}_1,\mathfrak{z}_2]$, and let $\mathscr{U}: J\subset\mathbb{R}\to\mathbb{R}$ and $\mathscr{W}:[\mathfrak{z}_1,\mathfrak{z}_2]\to\mathbb{R}$ be two distinct differentiable functions. Hence, we have*

$$\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\left(\mathscr{V}^*\left(\mathscr{U}(\ell)\right)^{\mathscr{U}'(\ell)\mathscr{W}(\ell)}\right)^{\mathrm{d}\ell}=\frac{\mathscr{V}(\mathscr{U}(\mathfrak{z}_2))^{\mathscr{W}(\mathfrak{z}_2)}}{\mathscr{V}(\mathscr{U}(\mathfrak{z}_1))^{\mathscr{W}(\mathfrak{z}_1)}}\times\frac{1}{\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\left(\mathscr{V}(\mathscr{U}(\ell))^{\mathscr{W}'(\ell)}\right)^{\mathrm{d}\ell}}.$$

Now, we recall the notion of multiplicative absolute value and its connection with the classical absolute value.

Definition 1.8 (see [7]). ***(Multiplicative absolute value)*** *Given $\mathscr{V}\in\mathbb{R}^+=(0,\infty)$, the multiplicative absolute value of $\mathscr{V}$, also called *-absolute value, is defined by*

$$|\mathscr{V}|_*=\begin{cases}\mathscr{V} & \text{if} \quad \mathscr{V}\geq 1\\ \frac{1}{\mathscr{V}} & \text{if} \quad \mathscr{V}<1.\end{cases}$$

The multiplicative absolute value $|.|_*$ and the classical absolute value $|.|$ are linked by the following relation

$$|\ln(\mathscr{V})|=\ln|\mathscr{V}|_*. \tag{1.1}$$

[see [7]]

Remark 1.9. ***(Multiplicative triangle inequality)*** *The multiplicative triangle inequality, which is also known as the *triangle inequality, is expressed as*

$$|\mathscr{V}\mathscr{W}|_*\leq|\mathscr{V}|_*|\mathscr{W}|_*,$$

for $\mathscr{V},\mathscr{W}\in\mathbb{R}^+$.

The counterpart to classical convexity within the framework of multiplicative calculus is referred to as logarithmic convexity or multiplicative convexity, which is defined as follows.

Definition 1.10 (see [8]). *A positive function $\mathscr{V}$ is considered multiplicative convex over the interval I if, for all $\ell_1,\ell_2\in I$, the following inequality*

$$\mathscr{V}(\mathfrak{l}\ell_1+(1-\mathfrak{l})\ell_2)\leq[\mathscr{V}(\ell_1)]^{\mathfrak{l}}[\mathscr{V}(\ell_2)]^{1-\mathfrak{l}},$$

holds true for all $\mathfrak{l}\in[0,1]$.

In the field of multiplicative integral inequalities, numerous researchers have focused on deriving error bounds for well-known quadrature formulas using multiplicative integral techniques. Particular attention has been paid to rules such as the midpoint and trapezoid formulas [9], the Ostrowski and Simpson rules [10], and Simpson second formula [11]. These studies contribute significantly to the understanding of approximation errors and inequality analysis within the framework of multiplicative calculus.

1.2. Fractional Calculus

While classical calculus deals with derivatives and integrals of integer order, fractional calculus extends these concepts to arbitrary (non-integer) orders. This generalization allows for a more flexible and nuanced description of dynamic systems, particularly those exhibiting memory effects or non-local behavior. The idea dates back to the late 17th century, with early contributions from mathematicians such as Leibniz and Liouville, and has since evolved into a powerful tool in applied mathematics, physics, engineering, and finance.

Over the years, various fractional-order operators have been introduced to capture different aspects of non-local dynamics. Among the most well-known are the Riemann–Liouville [12], Hadamard [13], and Katugampola [14]. Each of these formulations comes with specific properties that make them suitable for particular applications, depending on whether initial conditions, singular kernels, or integral averaging play a central role in the model under consideration.

Among the most celebrated integral operators are those of Riemann–Liouville, which are defined as:

Definition 1.11 (see [12]). *Let $\mathscr{V}\in L_1[\mathfrak{z}_1,\mathfrak{z}_2]$. The fractional Riemann-Liouville integrals $\mathscr{J}^{\gamma}_{\mathfrak{z}_1^+}\mathscr{V}$ and $\mathscr{J}^{\gamma}_{\mathfrak{z}_2^-}\mathscr{V}$ of order $\gamma>0$ with $\mathfrak{z}_1\geq 0$, are defined by*

$$\mathscr{J}^{\gamma}_{\mathfrak{z}_1^+}\mathscr{V}(\ell)=\frac{1}{\Gamma(\gamma)}\int_{\mathfrak{z}_1}^{\ell}(\ell-\mathfrak{l})^{\gamma-1}\mathscr{V}(\mathfrak{l})\mathrm{d}\mathfrak{l},\ \ell>\mathfrak{z}_1, \tag{1.2}$$

and

$$\mathscr{J}^{\gamma}_{\mathfrak{z}_2^-}\mathscr{V}(\ell)=\frac{1}{\Gamma(\gamma)}\int_{\ell}^{\mathfrak{z}_2}(\mathfrak{l}-\ell)^{\gamma-1}\mathscr{V}(\mathfrak{l})\mathrm{d}\mathfrak{l},\ \mathfrak{z}_2>\ell, \tag{1.3}$$

respectively, where $\Gamma(\nu)=\int_0^{\infty}e^{-u}u^{\nu-1}\,du$ is the gamma function, and $\mathscr{J}^{0}_{\mathfrak{z}_1^+}\mathscr{V}(\ell)=\mathscr{J}^{0}_{\mathfrak{z}_2^-}\mathscr{V}(\ell)=\mathscr{V}(\ell)$.

Several researchers have explored Milne's inequality within the framework of Riemann–Liouville fractional integrals, leading to a variety of results adapted to different classes of functions. These studies leverage the non-local nature of fractional operators to derive refined inequalities with broader applicability.

In [15], Budak and Hyder provided the following result related to Milne's inequality via Riemann-Liouville fractional integrals.

Theorem 1.12. *Let $\mathscr{V}:[\mathfrak{z}_1,\mathfrak{z}_2]\to\mathbb{R}$ be a differentiable mapping on $(\mathfrak{z}_1,\mathfrak{z}_2)$ such that $\mathscr{V}\in L_1([\mathfrak{z}_1,\mathfrak{z}_2])$. If $|\mathscr{V}'|$ is convex on $[\mathfrak{z}_1,\mathfrak{z}_2]$, then we have*

$$\begin{aligned}&\left|\frac{1}{3}\left(2\mathscr{V}(\mathfrak{z}_1)-\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)+2\mathscr{V}(\mathfrak{z}_2)\right)-\frac{\Gamma(\gamma+1)}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma}}\left(\mathscr{J}^{\gamma}_{\mathfrak{z}_1^+}\mathscr{V}(\mathfrak{z}_2)+\mathscr{J}^{\gamma}_{\mathfrak{z}_2^-}\mathscr{V}(\mathfrak{z}_1)\right)\right|\\ \leq&\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\mathscr{C}_1(\gamma)+\mathscr{C}_2(\gamma)\right)\left(\left|\mathscr{V}'(\mathfrak{z}_1)\right|+\left|\mathscr{V}'(\mathfrak{z}_2)\right|\right),\end{aligned}$$

where $\mathscr{J}^{\gamma}_{\mathfrak{z}_1^+}$ and $\mathscr{J}^{\gamma}_{\mathfrak{z}_2^-}$ are the left and right-hand sided Riemann-Liouville integrals, and $\mathscr{C}_1(\gamma)$ and $\mathscr{C}_2(\gamma)$ are given by

$$\mathscr{C}_1(\gamma)=\begin{cases}\dfrac{2\gamma}{\gamma+1}\left(\dfrac{2}{3}\right)^{\frac{1}{\gamma}+1}+\dfrac{1}{2^{\gamma+1}(\gamma+1)}-\dfrac{1}{3} & \text{for}\quad \left(\dfrac{2}{3}\right)^{\frac{1}{\gamma}}\leq\dfrac{1}{2}\\ \dfrac{1}{3}-\dfrac{1}{2^{\gamma+1}(\gamma+1)} & \text{for}\quad \left(\dfrac{2}{3}\right)^{\frac{1}{\gamma}}>\dfrac{1}{2},\end{cases}\tag{1.4}$$

and

$$\mathscr{C}_2(\gamma)=\begin{cases}\dfrac{1}{\gamma+1}-\dfrac{1}{2^{\gamma+1}(\gamma+1)}-\dfrac{1}{6} & \text{for}\quad \left(\dfrac{1}{3}\right)^{\frac{1}{\gamma}}\leq\dfrac{1}{2}\\ \dfrac{2\gamma}{\gamma+1}\left(\dfrac{1}{3}\right)^{\frac{1}{\gamma}+1}+\dfrac{1}{2^{\gamma+1}(\gamma+1)}+\dfrac{1}{\gamma+1}-\dfrac{1}{2} & \text{for}\quad \left(\dfrac{1}{3}\right)^{\frac{1}{\gamma}}>\dfrac{1}{2},\end{cases}\tag{1.5}$$

respectively.

Jarad et al. introduced conformable fractional derivatives and integrals in the following manner:

Definition 1.13 (see [16])**.** *Let $\mathscr{V}\in L_1[\mathfrak{z}_1,\mathfrak{z}_2]$. Then the conformable fractional integral operators of order $\gamma\in\mathbb{C}$ with $Re(\gamma)>0$ and $\omega\in(0,1]$ are defined by*

$${}_{\gamma}\mathscr{J}^{\omega}_{\mathfrak{z}_1^+}\mathscr{V}(\ell)=\frac{1}{\Gamma(\gamma)}\int_{\mathfrak{z}_1}^{\ell}\left(\frac{(\ell-\mathfrak{z}_1)^{\omega}-(\mathfrak{t}-\mathfrak{z}_1)^{\omega}}{\omega}\right)^{\gamma-1}(\mathfrak{t}-\mathfrak{z}_1)^{\omega-1}\mathscr{V}(\mathfrak{t})\mathrm{d}\mathfrak{t},\ \ell>\mathfrak{z}_1,\tag{1.6}$$

and

$${}_{\gamma}\mathscr{J}^{\omega}_{\mathfrak{z}_2^-}\mathscr{V}(\ell)=\frac{1}{\Gamma(\gamma)}\int_{\ell}^{\mathfrak{z}_2}\left(\frac{(\mathfrak{z}_2-\ell)^{\omega}-(\mathfrak{z}_2-\mathfrak{t})^{\omega}}{\omega}\right)^{\gamma-1}(\mathfrak{z}_2-\mathfrak{t})^{\omega-1}\mathscr{V}(\mathfrak{t})\mathrm{d}\mathfrak{t},\ \ell<\mathfrak{z}_2.\tag{1.7}$$

Remark 1.14. *In the case $\omega=1$, the conformable fractional integrals coincide with those of Riemann-Liouville.*

Several investigations have been conducted through the lens of inequalities via conformable fractional integrals. Notably, Set et al. advanced the understanding of Ostrowski-type inequalities in [17], later extending their analysis to Hermite–Hadamard-type settings in [18]. Further contributions include Kara et al., who focused on midpoint-type inequalities in [19], and Unal et al., who examined integral bounds for convex functions with applications to Newton-type expressions in [20]. More recently, Lakhdari et al. proposed a parameterized framework that facilitated the derivation of Hermite–Hadamard- and Maclaurin-type inequalities, as detailed in [21]. The topic of Milne-type inequalities has been explored in recent literature, notably by Demir [22], Hyder et al. [23], and Lakhdari et al. [24]. Collectively, these studies represent substantial progress in the field of fractional calculus, offering a diverse set of integral inequalities with broad implications in theoretical and applied mathematical analysis. Additional research addressing related topics can be found in [25–28].

We conclude this subsection by recalling the Beta and incomplete Beta functions, which are widely used in the field of fractional calculus.

Definition 1.15 (see [12])**.** *For any complex numbers and nonpositive integers ℓ_1,ℓ_2 such that $Re(\ell_1)>0$ and $Re(\ell_2)>0$. The Beta and incomplete Beta functions are defined by*

$$B(\ell_1,\ell_2)=\int_0^1\mathfrak{t}^{\ell_1-1}(1-\mathfrak{t})^{\ell_2-1}\,\mathrm{d}\mathfrak{t}=\frac{\Gamma(\ell_1)\Gamma(\ell_2)}{\Gamma(\ell_1+\ell_2)},$$

and

$$B_{\nu}(\ell_1,\ell_2)=\int_0^{\nu}\mathfrak{t}^{\ell_1-1}(1-\mathfrak{t})^{\ell_2-1}\,\mathrm{d}\mathfrak{t},\quad 0<\nu<1,$$

respectively, where $\Gamma(.)$ denotes the Euler gamma function.

1.3. Fractional Multiplicative Calculus

The concept of fractional multiplicative integrals and derivatives was first introduced by Abdeljawad and Grossman [29], merging the frameworks of fractional calculus and multiplicative analysis into a unified mathematical structure.

Definition 1.16 (see [29]). *The left-sided multiplicative Riemann-Liouville fractional integral* ${}_*\mathcal{J}^{\gamma}_{\mathfrak{z}_1^+}\mathcal{V}$ *of order* $\gamma\in\mathbb{C}, Re(\gamma)>0$ *is specified by*

$$ {}_*\mathcal{J}^{\gamma}_{\mathfrak{z}_1^+}\mathcal{V}(\ell)=\exp\left\{\left(\mathcal{J}^{\gamma}_{\mathfrak{z}_1^+}(\ln\circ\mathcal{V})\right)(\ell)\right\},\ \ell>\mathfrak{z}_1, $$

and the right-sided one is defined by

$$ {}_*\mathcal{J}^{\gamma}_{\mathfrak{z}_2^-}\mathcal{V}(\ell)=\exp\left\{\left(\mathcal{J}^{\gamma}_{\mathfrak{z}_2^-}(\ln\circ\mathcal{V})\right)(\ell)\right\},\ \ell<\mathfrak{z}_2, $$

where $\mathcal{J}^{\gamma}_{\mathfrak{z}_1^+}$ *and* $\mathcal{J}^{\gamma}_{\mathfrak{z}_2^-}$ *denote the left and right Riemann-Liouville fractional integrals defined as in (1.2) and (1.3), respectively.*

Following the introduction of fractional multiplicative integrals, a significant body of research has emerged focusing on the development of various integral inequalities within this framework. Scholars have investigated extensions of classical inequalities such as Ostrowski-type [30], Simpson-type [31], corrected dual-Simpson-type [32], Bullen-type [33], Maclaurin-type [34,35], and Newton-type inequalities [36]. These studies contribute to a growing collection of results that adapt well-known analytical tools to the fractional multiplicative setting. Further related contributions can be found in [37,38], along with the references listed therein.

In their work [39], Almatrafi et al. investigated Riemann–Liouville fractional multiplicative integral inequalities using a multiparameter approach, leading to the derivation of a new inequality related to Milne's quadrature rule.

Theorem 1.17. *Let* $\mathcal{V}:[\mathfrak{z}_1,\mathfrak{z}_2]\to\mathbb{R}^+$ *be a* **-increasing and* **-differentiable function over* $[\mathfrak{z}_1,\mathfrak{z}_2]$. *If* $\mathcal{V}^*$ *is multiplicative s-convex on this interval, then for any* $\gamma>0$, *the following holds:*

$$ \left|\frac{\left((\mathcal{V}(\mathfrak{z}_1))^2\left(\mathcal{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)^{-1}(\mathcal{V}(\mathfrak{z}_2))^2\right)^{\frac{1}{3}}}{\left[\left({}_*\mathcal{J}^{\gamma}_{\mathfrak{z}_1^+}\mathcal{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)\left({}_*\mathcal{J}^{\gamma}_{\mathfrak{z}_2^-}\mathcal{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)\right]^{\frac{2^{\gamma-1}\Gamma(\gamma+1)}{(\mathfrak{z}_2-\mathfrak{z}_1)^{2\gamma-1}}}}\right|_*\le\left[(\mathcal{V}^*(\mathfrak{z}_1)\mathcal{V}^*(\mathfrak{z}_2))^{\frac{\gamma+4s+4}{3(s+1)(\gamma+s+1)}}\mathcal{V}^*\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)^{2\left(\frac{1}{3(s+1)}+B(s+1,\gamma+1)\right)}\right]^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{4}}, $$

where $\mathcal{V}^*$ *is multiplicative s-convex in the second sense, and* $B(.,.)$ *is the Beta function.*

In recent years, several novel multiplicative fractional operators have been proposed, including the multiplicative conformable fractional integrals introduced in 2025 by Budak and Ergün [40]. They subsequently established fundamental properties of these newly defined integrals, such as integrability, boundedness, continuity, and the semigroup property.

Definition 1.18 (see [40]). *The left-sided conformable fractional multiplicative integral* ${}^{\gamma}_{\mathfrak{z}_1}\mathcal{I}^{\omega}_{*}\mathcal{V}(\ell)$ *of order* $\gamma\in\mathbb{C}$, $Re(\gamma)>0$ *and* $\omega\in(0,1]$, *is defined by*

$$ {}^{\gamma}_{\mathfrak{z}_1}\mathcal{I}^{\omega}_{*}\mathcal{V}(\ell)=\exp\left\{\left({}^{\gamma}_{\mathfrak{z}_1}\mathcal{I}^{\omega}(ln\circ\mathcal{V})\right)(\ell)\right\},\ \ell>\mathfrak{z}_1, \tag{1.8} $$

and the right-sided one ${}^{\gamma}_{*}\mathcal{I}^{\omega}_{\mathfrak{z}_2}\mathcal{V}(\ell)$ *is defined by*

$$ {}^{\gamma}_{*}\mathcal{I}^{\omega}_{\mathfrak{z}_2}\mathcal{V}(\ell)=\exp\left\{\left({}^{\gamma}\mathcal{I}^{\omega}_{\mathfrak{z}_2}(ln\circ\mathcal{V})\right)(\ell)\right\},\ \ell<\mathfrak{z}_2, \tag{1.9} $$

where the symbols ${}^{\gamma}\mathcal{I}^{\omega}_{\mathfrak{z}_1^+}\mathcal{V}(\ell)$ *and* ${}^{\gamma}\mathcal{I}^{\omega}_{\mathfrak{z}_2^-}\mathcal{V}(\ell)$ *denote the left-sided and right-sided conformable fractional integrals defined as in (1.6) and (1.7), respectively.*

Subsequently, the authors established Hermite–Hadamard inequalities in three distinct senses for these conformable fractional multiplicative integrals.

Theorem 1.19 (see [40]). *Suppose that the mapping* $\mathcal{V}$ *is positive and multiplicative convex on the interval* $[\mathfrak{z}_1,\mathfrak{z}_2]$. *Then, for* $\gamma>0$ *and* $\omega\in(0,1]$, *we have*

$$ \mathcal{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\le\left[\left({}^{\gamma}_{\mathfrak{z}_1}\mathcal{I}^{\omega}_{*}\mathcal{V}(\mathfrak{z}_2)\right)\left({}^{\gamma}_{*}\mathcal{I}^{\omega}_{\mathfrak{z}_2}\mathcal{V}(\mathfrak{z}_1)\right)\right]^{\frac{\Gamma(\gamma+1)\omega^{\gamma}}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}\le\sqrt{\mathcal{V}(\mathfrak{z}_1)\mathcal{V}(\mathfrak{z}_2)}, $$

$$ \mathcal{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\le\left[\left({}^{\gamma}_{\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)}\mathcal{I}^{\omega}_{*}\mathcal{V}(\mathfrak{z}_2)\right)\left({}^{\gamma}_{*}\mathcal{I}^{\omega}_{\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)}\mathcal{V}(\mathfrak{z}_1)\right)\right]^{\frac{\Gamma(\gamma+1)\omega^{\gamma}2^{\gamma\omega-1}}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}\le\sqrt{\mathcal{V}(\mathfrak{z}_1)\mathcal{V}(\mathfrak{z}_2)}, $$

and

$$ \mathcal{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\le\left[\left({}^{\gamma}_{\mathfrak{z}_1}\mathcal{I}^{\omega}_{*}\mathcal{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)\left({}^{\gamma}_{*}\mathcal{I}^{\omega}_{\mathfrak{z}_2}\mathcal{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)\right]^{\frac{\Gamma(\gamma+1)\omega^{\gamma}2^{\gamma\omega-1}}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}\le\sqrt{\mathcal{V}(\mathfrak{z}_1)\mathcal{V}(\mathfrak{z}_2)}. $$

Furthermore, they presented several corresponding midpoint and trapezoidal inequalities associated with the obtained Hermite–Hadamard inequalities, which involve conformable fractional multiplicative integrals.

Theorem 1.20 (see [40]). *Let $\mathscr{V}:[\mathfrak{z}_1,\mathfrak{z}_2]\to\mathbb{R}^+$ be a $*$-differentiable function on $(\mathfrak{z}_1,\mathfrak{z}_2)$. Under the hypothesis that $\mathscr{V}^*$ possesses multiplicative convexity on $[\mathfrak{z}_1,\mathfrak{z}_2]$ together with $\gamma>0$, $\omega\in(0,1]$ and $\mathscr{V}^*\geq 1$, then the following midpoint-type inequalities hold*

$$\frac{\left[\left({}^{\gamma}_{\mathfrak{z}_1}\mathscr{I}^{\omega}_{*}\mathscr{V}(\mathfrak{z}_2)\right)\left({}^{\gamma}_{*}\mathscr{I}^{\omega}_{\mathfrak{z}_2}\mathscr{V}(\mathfrak{z}_1)\right)\right]^{\frac{\Gamma(\gamma+1)\omega^{\gamma}}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}}{\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)}\leq\left[\mathscr{V}^*(\mathfrak{z}_1)\mathscr{V}^*(\mathfrak{z}_2)\right]^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2\omega}\left[\mathrm{B}\left(\frac{1}{\omega},\gamma+1\right)+\frac{\omega}{2}-2\mathrm{B}_{\left(\frac{1}{2}\right)^{\gamma}}\left(\frac{1}{\omega},\gamma+1\right)\right]},$$

$$\frac{\left[\left({}^{\gamma}_{\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)}\mathscr{I}^{\omega}_{*}\mathscr{V}(\mathfrak{z}_2)\right)\left({}^{\gamma}_{*}\mathscr{I}^{\omega}_{\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)}\mathscr{V}(\mathfrak{z}_1)\right)\right]^{\frac{\Gamma(\gamma+1)\omega^{\gamma}2^{\gamma\omega-1}}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}}{\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)}\leq\left[\mathscr{V}^*(\mathfrak{z}_1)\mathscr{V}^*(\mathfrak{z}_2)\right]^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{4\omega}\mathrm{B}\left(\frac{1}{\omega},\gamma+1\right)},$$

and

$$\frac{\left[\left({}^{\gamma}_{\mathfrak{z}_1}\mathscr{I}^{\omega}_{*}\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)\left({}^{\gamma}_{*}\mathscr{I}^{\omega}_{\mathfrak{z}_2}\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)\right]^{\frac{\Gamma(\gamma+1)\omega^{\gamma}2^{\gamma\omega-1}}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}}{\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)}\leq\left[\mathscr{V}^*(\mathfrak{z}_1)\mathscr{V}^*(\mathfrak{z}_2)\right]^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{4\omega}\left[\omega-\mathrm{B}\left(\frac{1}{\omega},\gamma+1\right)\right]},$$

where $\mathrm{B}(.,.)$ and $\mathrm{B}_{\sigma}(.,.)$ represent the beta and incomplete beta function, respectively.

Theorem 1.21 (see [40]). *Let $\mathscr{V}:[\mathfrak{z}_1,\mathfrak{z}_2]\to\mathbb{R}^+$ be a $*$-differentiable function on $(\mathfrak{z}_1,\mathfrak{z}_2)$. Under the hypothesis that $\mathscr{V}$ is $*$-increasing and $\mathscr{V}^*$ possesses multiplicative convexity on $[\mathfrak{z}_1,\mathfrak{z}_2]$ together with $\gamma>0$, $\omega\in(0,1]$ and $\mathscr{V}^*\geq 1$, then the following trapezoid-type inequalities hold*

$$\frac{\sqrt{\mathscr{V}(\mathfrak{z}_1)\mathscr{V}(\mathfrak{z}_2)}}{\left[\left({}^{\gamma}_{\mathfrak{z}_1}\mathscr{I}^{\omega}_{*}\mathscr{V}(\mathfrak{z}_2)\right)\left({}^{\gamma}_{*}\mathscr{I}^{\omega}_{\mathfrak{z}_2}\mathscr{V}(\mathfrak{z}_1)\right)\right]^{\frac{\Gamma(\gamma+1)\omega^{\gamma}}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}}\leq\left[\mathscr{V}^*(\mathfrak{z}_1)\mathscr{V}^*(\mathfrak{z}_2)\right]^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left[2\mathrm{B}_{\left(\frac{1}{2}\right)^{\gamma}}\left(\frac{1}{\omega},\gamma+1\right)-\mathrm{B}\left(\frac{1}{\omega},\gamma+1\right)\right]},$$

$$\frac{\sqrt{\mathscr{V}(\mathfrak{z}_1)\mathscr{V}(\mathfrak{z}_2)}}{\left[\left({}^{\gamma}_{\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)}\mathscr{I}^{\omega}_{*}\mathscr{V}(\mathfrak{z}_2)\right)\left({}^{\gamma}_{*}\mathscr{I}^{\omega}_{\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)}\mathscr{V}(\mathfrak{z}_1)\right)\right]^{\frac{\Gamma(\gamma+1)\omega^{\gamma}2^{\gamma\omega-1}}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}}\leq\left[\mathscr{V}^*(\mathfrak{z}_1)\mathscr{V}^*(\mathfrak{z}_2)\right]^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{4\omega}\left[\omega-\mathrm{B}\left(\frac{1}{\omega},\gamma+1\right)\right]},$$

and

$$\frac{\sqrt{\mathscr{V}(\mathfrak{z}_1)\mathscr{V}(\mathfrak{z}_2)}}{\left[\left({}^{\gamma}_{\mathfrak{z}_1}\mathscr{I}^{\omega}_{*}\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)\left({}^{\gamma}_{*}\mathscr{I}^{\omega}_{\mathfrak{z}_2}\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)\right]^{\frac{\Gamma(\gamma+1)\omega^{\gamma}2^{\gamma\omega-1}}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}}\leq\left[\mathscr{V}^*(\mathfrak{z}_1)\mathscr{V}^*(\mathfrak{z}_2)\right]^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{4\omega}\mathrm{B}\left(\frac{1}{\omega},\gamma+1\right)},$$

where $\mathrm{B}(.,.)$ and $\mathrm{B}_{\sigma}(.,.)$ represent the beta and incomplete beta function, respectively.

Inspired by the ongoing advancements in multiplicative fractional calculus, this paper aims to investigate Milne's inequality in the context of newly defined conformable fractional multiplicative integrals. To achieve this objective, we introduce a new identity in Section 2, which serves as a key tool for deriving a Milne-type inequality applicable to two classes of functions: those with *-derivatives in $*$absolute value possessing the multiplicative convexity property, and those with bounded *-derivatives. These results are presented in Section 3, and are followed by a numerical example, supported by graphical illustrations, to demonstrate the validity and precision of our theoretical findings. Further results concerning functions whose *-derivatives satisfy the condition $(\ln\mathscr{V}^*)^m$ $(m>1)$ are discussed in Section 4. The study concludes with an application to special means, detailed in Section 5, followed by a concluding section (Section 6) that outlines key contributions and suggests potential directions for future research.

2. A New Identity in Conformable Fractional Multiplicative Calculus

To lay the groundwork for deriving Milne-type inequalities in the context of conformable fractional multiplicative integrals, this section introduces a novel identity involving *-differentiable functions. This identity will serve as a crucial tool in the subsequent error analysis and inequality derivation.

Lemma 2.1. *Let $\mathscr{V}:[\mathfrak{z}_1,\mathfrak{z}_2]\to\mathbb{R}^+$ be a *-differentiable function on $[\mathfrak{z}_1,\mathfrak{z}_2]$ with $0\leq\mathfrak{z}_1<\mathfrak{z}_2$, and $\mathscr{V}^*\in L^1[\mathfrak{z}_1,\mathfrak{z}_2]$. Then, for $\gamma>0$ and $\omega\in(0,1]$, the following equality holds*

$$\begin{aligned}
&\frac{\left[(\mathscr{V}(\mathfrak{z}_1))^2\left(\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)^{-1}(\mathscr{V}(\mathfrak{z}_2))^2\right]^{\frac{1}{3\omega^{\gamma}}}}{\left[\left({}^{\gamma}_{\mathfrak{z}_1}\mathscr{I}^{\omega}_{*}\mathscr{V}\right)(\mathfrak{z}_2)\left({}^{\gamma}_{*}\mathscr{I}^{\omega}_{\mathfrak{z}_2}\mathscr{V}\right)(\mathfrak{z}_1)\right]^{\frac{\Gamma(\gamma+1)}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}}\\
&=\left(\int_0^{\frac{1}{2}}\left((\mathscr{V}^*((1-\mathfrak{l})\mathfrak{z}_1+\mathfrak{l}\mathfrak{z}_2))^{\left(\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right)}\right)\mathrm{d}\mathfrak{l}\right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}}\left(\int_{\frac{1}{2}}^1\left((\mathscr{V}^*((1-\mathfrak{l})\mathfrak{z}_1+\mathfrak{l}\mathfrak{z}_2))^{\left(\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right)}\right)\mathrm{d}\mathfrak{l}\right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}}\\
&\times\left(\int_0^{\frac{1}{2}}\left((\mathscr{V}^*(\mathfrak{l}\mathfrak{z}_1+(1-\mathfrak{l})\mathfrak{z}_2))^{\left(\frac{2}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right)}\right)\mathrm{d}\mathfrak{l}\right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}}\left(\int_{\frac{1}{2}}^1\left((\mathscr{V}^*(\mathfrak{l}\mathfrak{z}_1+(1-\mathfrak{l})\mathfrak{z}_2))^{\left(\frac{1}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right)}\right)\mathrm{d}\mathfrak{l}\right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}},
\end{aligned}$$

where ${}^{\gamma}_{\mathfrak{z}_1}\mathscr{I}^{\omega}_{*}$ and ${}^{\gamma}_{*}\mathscr{I}^{\omega}_{\mathfrak{z}_2}$ are the left and right conformable fractional multiplicative integrals given by (1.8) and (1.9), respectively.

Proof. Let

$$\mathscr{E}_1 = \left(\int_0^{\frac{1}{2}} \left((\mathscr{V}^*((1-\mathfrak{l})\mathfrak{z}_1 + \mathfrak{l}\mathfrak{z}_2))^{\left(\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma} - \frac{2}{3\omega^{\gamma}}\right)} \right) \mathrm{d}\mathfrak{l} \right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}},$$

$$\mathscr{E}_2 = \left(\int_{\frac{1}{2}}^{1} \left((\mathscr{V}^*((1-\mathfrak{l})\mathfrak{z}_1 + \mathfrak{l}\mathfrak{z}_2))^{\left(\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma} - \frac{1}{3\omega^{\gamma}}\right)} \right) \mathrm{d}\mathfrak{l} \right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}},$$

$$\mathscr{E}_3 = \left(\int_0^{\frac{1}{2}} \left((\mathscr{V}^*(\mathfrak{l}\mathfrak{z}_1 + (1-\mathfrak{l})\mathfrak{z}_2))^{\left(\frac{2}{3\omega^{\gamma}} - \left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right)} \right) \mathrm{d}\mathfrak{l} \right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}}$$

and

$$\mathscr{E}_4 = \left(\int_{\frac{1}{2}}^{1} \left((\mathscr{V}^*(\mathfrak{l}\mathfrak{z}_1 + (1-\mathfrak{l})\mathfrak{z}_2))^{\left(\frac{1}{3\omega^{\gamma}} - \left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right)} \right) \mathrm{d}\mathfrak{l} \right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}}.$$

Integrating by parts $\mathscr{E}_1$, we get

$$\begin{aligned}
\mathscr{E}_1 &= \left(\int_0^{\frac{1}{2}} \left((\mathscr{V}^*(\mathfrak{l}\mathfrak{z}_1 + (1-\mathfrak{l})\mathfrak{z}_2))^{\left(\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma} - \frac{2}{3\omega^{\gamma}}\right)} \right) \mathrm{d}\mathfrak{l} \right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}} \\
&= \exp\left\{ \int_0^{\frac{1}{2}} \left(\ln\left(\mathscr{V}^*((1-\mathfrak{l})\mathfrak{z}_1 + \mathfrak{l}\mathfrak{z}_2)\right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma} - \frac{2}{3\omega^{\gamma}}\right)} \right) \mathrm{d}\mathfrak{l} \right\} \\
&= \exp\left\{ \int_0^{\frac{1}{2}} \left(\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2} \left(\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma} - \frac{2}{3\omega^{\gamma}} \right) \ln\left(\mathscr{V}^*((1-\mathfrak{l})\mathfrak{z}_1 + \mathfrak{l}\mathfrak{z}_2)\right) \right) \mathrm{d}\mathfrak{l} \right\} \\
&= \exp\left\{ \int_0^{\frac{1}{2}} \frac{\mathfrak{z}_2-\mathfrak{z}_1}{2} \left(\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma} - \frac{2}{3\omega^{\gamma}} \right) \left(\ln \mathscr{V}((1-\mathfrak{l})\mathfrak{z}_1 + \mathfrak{l}\mathfrak{z}_2)\right)' \mathrm{d}\mathfrak{l} \right\} \\
&= \exp\left\{ \frac{1}{2} \left(\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma} - \frac{2}{3\omega^{\gamma}} \right) \ln\left(\mathscr{V}((1-\mathfrak{l})\mathfrak{z}_1 + \mathfrak{l}\mathfrak{z}_2)\right) \Big|_0^{\frac{1}{2}} \right. \\
&\quad \left. - \frac{\gamma}{2} \int_0^{\frac{1}{2}} \left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma-1} (1-\mathfrak{l})^{\omega-1} \ln\left(\mathscr{V}((1-\mathfrak{l})\mathfrak{z}_1 + \mathfrak{l}\mathfrak{z}_2)\right) \mathrm{d}\mathfrak{l} \right\} \\
&= \exp\left\{ \frac{1}{2} \left(\left(\frac{1-\left(\frac{1}{2}\right)^{\omega}}{\omega}\right)^{\gamma} - \frac{2}{3\omega^{\gamma}} \right) \left(\ln \mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right) \right) + \frac{1}{3\omega^{\gamma}} \left(\ln \mathscr{V}(\mathfrak{z}_1)\right) \right. \\
&\quad \left. - \int_0^{\frac{1}{2}} \frac{\gamma}{2} \left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma-1} (1-\mathfrak{l})^{\omega-1} \ln\left(\mathscr{V}((1-\mathfrak{l})\mathfrak{z}_1 + \mathfrak{l}\mathfrak{z}_2)\right) \mathrm{d}\mathfrak{l} \right\} \\
&= \exp\left\{ \frac{1}{2} \left(\left(\frac{1-\left(\frac{1}{2}\right)^{\omega}}{\omega}\right)^{\gamma} - \frac{2}{3\omega^{\gamma}} \right) \left(\ln \mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right) \right) + \frac{1}{3\omega^{\gamma}} \left(\ln \mathscr{V}(\mathfrak{z}_1)\right) \right. \\
&\quad \left. - \frac{\Gamma(\gamma+1)}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}} \left(\frac{1}{\Gamma(\gamma)} \int_{\mathfrak{z}_1}^{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}} \left(\frac{(\mathfrak{z}_2-\mathfrak{z}_1)^{\omega} - (\mathfrak{z}_2-\ell)^{\omega}}{\omega} \right)^{\gamma-1} (\mathfrak{z}_2-\ell)^{\omega-1} \ln\left(\mathscr{V}(\ell)\right) \mathrm{d}\ell \right) \right\}.
\end{aligned} \tag{2.1}$$

Similarly, we obtain

$$\mathscr{E}_2 = \left(\int_{\frac{1}{2}}^{1} \left((\mathscr{V}^*((1-\mathfrak{l})\mathfrak{z}_1 + \mathfrak{l}\mathfrak{z}_2))^{\left(\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma} - \frac{1}{3\omega^{\gamma}}\right)} \right) \mathrm{d}\mathfrak{l} \right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}}$$

$$=\exp\left\{\frac{1}{3\omega^{\gamma}}\left(\ln\mathscr{V}\left(\mathfrak{z}_2\right)\right)-\frac{1}{2}\left(\left(\frac{1-\left(\frac{1}{2}\right)^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right)\left(\ln\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)\right. \tag{2.2}$$

$$\left.-\frac{\Gamma(\gamma+1)}{2\left(\mathfrak{z}_2-\mathfrak{z}_1\right)^{\gamma\omega}}\left(\frac{1}{\Gamma(\gamma)}\int_{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}}^{\mathfrak{z}_2}\left(\frac{\left(\mathfrak{z}_2-\mathfrak{z}_1\right)^{\omega}-\left(\mathfrak{z}_2-\ell\right)^{\omega}}{\omega}\right)^{\gamma-1}\left(\mathfrak{z}_2-\ell\right)^{\omega-1}\ln\left(\mathscr{V}\left(\ell\right)\right)\mathrm{d}\ell\right)\right\},$$

$$\mathscr{E}_3=\left(\int_0^{\frac{1}{2}}\left(\left(\mathscr{V}^*\left(\mathfrak{l}\mathfrak{z}_1+\left(1-\mathfrak{l}\right)\mathfrak{z}_2\right)\right)^{\left(\frac{2}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right)}\right)\mathrm{d}\mathfrak{l}\right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}}$$

$$=\exp\left\{\frac{1}{2}\left(\left(\frac{1-\left(\frac{1}{2}\right)^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right)\left(\ln\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)+\frac{1}{3\omega^{\gamma}}\left(\ln\mathscr{V}\left(\mathfrak{z}_2\right)\right)\right. \tag{2.3}$$

$$\left.+\frac{\Gamma(\gamma+1)}{2\left(\mathfrak{z}_2-\mathfrak{z}_1\right)^{\gamma\omega}}\left(\frac{1}{\Gamma(\gamma)}\int_{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}}^{\mathfrak{z}_2}\left(\frac{\left(\mathfrak{z}_2-\mathfrak{z}_1\right)^{\omega}-\left(\ell-\mathfrak{z}_1\right)^{\omega}}{\omega}\right)^{\gamma-1}\left(\ell-\mathfrak{z}_1\right)^{\omega-1}\ln\left(\mathscr{V}\left(\ell\right)\right)\mathrm{d}\ell\right)\right\}$$

and

$$\mathscr{E}_4=\left(\int_{\frac{1}{2}}^{1}\left(\left(\mathscr{V}^*\left(\mathfrak{l}\mathfrak{z}_1+\left(1-\mathfrak{l}\right)\mathfrak{z}_2\right)\right)^{\left(\frac{1}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right)}\right)\mathrm{d}\mathfrak{l}\right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}}$$

$$=\exp\left\{\frac{1}{3\omega^{\gamma}}\left(\ln\mathscr{V}\left(\mathfrak{z}_1\right)\right)+\frac{1}{2}\left(\frac{1}{3\omega^{\gamma}}-\left(\frac{1-\left(\frac{1}{2}\right)^{\omega}}{\omega}\right)^{\gamma}\right)\left(\ln\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)\right. \tag{2.4}$$

$$\left.-\frac{\Gamma(\gamma+1)}{2\left(\mathfrak{z}_2-\mathfrak{z}_1\right)^{\gamma\omega}}\left(\frac{1}{\Gamma(\gamma)}\int_{\mathfrak{z}_1}^{\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}}\left(\frac{\left(\mathfrak{z}_2-\mathfrak{z}_1\right)^{\omega}-\left(\ell-\mathfrak{z}_1\right)^{\omega}}{\omega}\right)^{\gamma-1}\left(\ell-\mathfrak{z}_1\right)^{\omega-1}\ln\left(\mathscr{V}\left(\ell\right)\right)\mathrm{d}\ell\right)\right\}.$$

□

Finally, the desired result is obtained by multiplying (2.1)-(2.4).

Thus, the proof is completed.

3. Milne-Type Inequalities via Conformable Fractional Multiplicative Integrals

Building on the identity established in the previous section, this part focuses on deriving a Milne-type inequality for functions whose *-derivatives in $*$absolute value satisfy a multiplicative convexity condition as well as functions with bounded *-derivatives.

3.1. Milne-Type Inequality Under Multiplicative Convexity

Theorem 3.1. *Let $\mathscr{V}:[\mathfrak{z}_1,\mathfrak{z}_2]\to\mathbb{R}^+$ be a *-differentiable mapping on $[\mathfrak{z}_1,\mathfrak{z}_2]$ with $0\le\mathfrak{z}_1<\mathfrak{z}_2$. If $|\mathscr{V}^*|_*$ is multiplicative convex on $[\mathfrak{z}_1,\mathfrak{z}_2]$, then the inequality holds*

$$\left|\frac{\left[\left(\mathscr{V}\left(\mathfrak{z}_1\right)\right)^2\left(\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)^{-1}\left(\mathscr{V}\left(\mathfrak{z}_2\right)\right)^2\right]^{\frac{1}{3\omega^{\gamma}}}}{\left[\left({}_{\mathfrak{z}_1}^{\gamma}\mathscr{J}_*^{\omega}\mathscr{V}\right)\left(\mathfrak{z}_2\right)\left({}_*^{\gamma}\mathscr{J}_{\mathfrak{z}_2}^{\omega}\mathscr{V}\right)\left(\mathfrak{z}_1\right)\right]^{\frac{\Gamma(\gamma+1)}{2\left(\mathfrak{z}_2-\mathfrak{z}_1\right)^{\gamma\omega}}}}\right|_*\le\left(\left|\mathscr{V}^*\left(\mathfrak{z}_1\right)\right|_*\left|\mathscr{V}^*\left(\mathfrak{z}_2\right)\right|_*\right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\mathscr{A}_1(\gamma,\omega)+\mathscr{A}_2(\gamma,\omega)\right)},$$

where $\mathrm{B}(.,.)$ and $\mathrm{B}_{\sigma}(.,.)$ are the Beta and the incomplete Beta functions, respectively, and $\mathscr{A}_1$ and $\mathscr{A}_2$ are defined as in (3.2) and (3.3).

Proof. By applying the *-absolute value on both sides of the inequality given in Lemma 2.1 and using relation (1.1), we obtain

$$\left|\frac{\left[\left(\mathscr{V}\left(\mathfrak{z}_1\right)\right)^2\left(\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)^{-1}\left(\mathscr{V}\left(\mathfrak{z}_2\right)\right)^2\right]^{\frac{1}{3\omega^{\gamma}}}}{\left[\left({}_{\mathfrak{z}_1}^{\gamma}\mathscr{J}_*^{\omega}\mathscr{V}\right)\left(\mathfrak{z}_2\right)\left({}_*^{\gamma}\mathscr{J}_{\mathfrak{z}_2}^{\omega}\mathscr{V}\right)\left(\mathfrak{z}_1\right)\right]^{\frac{\Gamma(\gamma+1)}{2\left(\mathfrak{z}_2-\mathfrak{z}_1\right)^{\gamma\omega}}}}\right|_*$$

$$\le\left|\left(\int_0^{\frac{1}{2}}\left(\left(\mathscr{V}^*\left(\left(1-\mathfrak{l}\right)\mathfrak{z}_1+\mathfrak{l}\mathfrak{z}_2\right)\right)^{\left(\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right)}\right)\mathrm{d}\mathfrak{l}\right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}}\right|_*$$

$$
\begin{aligned}
&\times\left|\left(\int_{\frac{1}{2}}^{1}\left(\left(\mathscr{V}^{*}\left(\left(1-\mathfrak{l}\right)\mathfrak{z}_{1}+\mathfrak{l}\mathfrak{z}_{2}\right)\right)^{\left(\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right)}\right)\mathrm{d}\mathfrak{l}\right)^{\frac{\mathfrak{z}_{2}-\mathfrak{z}_{1}}{2}}\right|_{*}\\
&\times\left|\left(\int_{0}^{\frac{1}{2}}\left(\left(\mathscr{V}^{*}\left(\mathfrak{l}\mathfrak{z}_{1}+\left(1-\mathfrak{l}\right)\mathfrak{z}_{2}\right)\right)^{\left(\frac{2}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right)}\right)\mathrm{d}\mathfrak{l}\right)^{\frac{\mathfrak{z}_{2}-\mathfrak{z}_{1}}{2}}\right|_{*}\\
&\times\left|\left(\int_{\frac{1}{2}}^{1}\left(\left(\mathscr{V}^{*}\left(\mathfrak{l}\mathfrak{z}_{1}+\left(1-\mathfrak{l}\right)\mathfrak{z}_{2}\right)\right)^{\left(\frac{1}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right)}\right)\mathrm{d}\mathfrak{l}\right)^{\frac{\mathfrak{z}_{2}-\mathfrak{z}_{1}}{2}}\right|_{*}\\
&\leq\exp\left\{\frac{\mathfrak{z}_{2}-\mathfrak{z}_{1}}{2}\int_{0}^{\frac{1}{2}}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right|\left|\ln\left(\mathscr{V}^{*}\left(\left(1-\mathfrak{l}\right)\mathfrak{z}_{1}+\mathfrak{l}\mathfrak{z}_{2}\right)\right)\right|\mathrm{d}\mathfrak{l}\right\}\\
&\times\exp\left\{\frac{\mathfrak{z}_{2}-\mathfrak{z}_{1}}{2}\int_{\frac{1}{2}}^{1}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|\left|\ln\left(\mathscr{V}^{*}\left(\left(1-\mathfrak{l}\right)\mathfrak{z}_{1}+\mathfrak{l}\mathfrak{z}_{2}\right)\right)\right|\mathrm{d}\mathfrak{l}\right\}\\
&\times\exp\left\{\frac{\mathfrak{z}_{2}-\mathfrak{z}_{1}}{2}\int_{0}^{\frac{1}{2}}\left|\frac{2}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|\left|\ln\left(\mathscr{V}^{*}\left(\mathfrak{l}\mathfrak{z}_{1}+\left(1-\mathfrak{l}\right)\mathfrak{z}_{2}\right)\right)\right|\mathrm{d}\mathfrak{l}\right\}\\
&\times\exp\left\{\frac{\mathfrak{z}_{2}-\mathfrak{z}_{1}}{2}\int_{\frac{1}{2}}^{1}\left|\frac{1}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|\left|\ln\left(\mathscr{V}^{*}\left(\mathfrak{l}\mathfrak{z}_{1}+\left(1-\mathfrak{l}\right)\mathfrak{z}_{2}\right)\right)\right|\mathrm{d}\mathfrak{l}\right\}\\
&=\exp\left\{\frac{\mathfrak{z}_{2}-\mathfrak{z}_{1}}{2}\int_{0}^{\frac{1}{2}}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right|\left|\ln\left|\mathscr{V}^{*}\left(\left(1-\mathfrak{l}\right)\mathfrak{z}_{1}+\mathfrak{l}\mathfrak{z}_{2}\right)\right|_{*}\right|\mathrm{d}\mathfrak{l}\right\}\\
&\times\exp\left\{\frac{\mathfrak{z}_{2}-\mathfrak{z}_{1}}{2}\int_{\frac{1}{2}}^{1}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|\left|\ln\left|\mathscr{V}^{*}\left(\left(1-\mathfrak{l}\right)\mathfrak{z}_{1}+\mathfrak{l}\mathfrak{z}_{2}\right)\right|_{*}\right|\mathrm{d}\mathfrak{l}\right\}\\
&\times\exp\left\{\frac{\mathfrak{z}_{2}-\mathfrak{z}_{1}}{2}\int_{0}^{\frac{1}{2}}\left|\frac{2}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|\left|\ln\left|\mathscr{V}^{*}\left(\mathfrak{l}\mathfrak{z}_{1}+\left(1-\mathfrak{l}\right)\mathfrak{z}_{2}\right)\right|_{*}\right|\mathrm{d}\mathfrak{l}\right\}\\
&\times\exp\left\{\frac{\mathfrak{z}_{2}-\mathfrak{z}_{1}}{2}\int_{\frac{1}{2}}^{1}\left|\frac{1}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|\left|\ln\left|\mathscr{V}^{*}\left(\mathfrak{l}\mathfrak{z}_{1}+\left(1-\mathfrak{l}\right)\mathfrak{z}_{2}\right)\right|_{*}\right|\mathrm{d}\mathfrak{l}\right\}. \qquad (3.1)
\end{aligned}
$$

By making use of the multiplicative convexity of $|\mathscr{V}^{*}|_{*}$, inequality (3.1) yields

$$
\begin{aligned}
&\left|\frac{\left[\left(\mathscr{V}\left(\mathfrak{z}_{1}\right)\right)^{2}\left(\mathscr{V}\left(\frac{\mathfrak{z}_{1}+\mathfrak{z}_{2}}{2}\right)\right)^{-1}\left(\mathscr{V}\left(\mathfrak{z}_{2}\right)\right)^{2}\right]^{\frac{1}{3\omega^{\gamma}}}}{\left[\left({}_{\mathfrak{z}_{1}}^{\gamma}\mathscr{J}_{*}^{\omega}\mathscr{V}\right)\left(\mathfrak{z}_{2}\right)\left({}_{*}^{\gamma}\mathscr{J}_{\mathfrak{z}_{2}}^{\omega}\mathscr{V}\right)\left(\mathfrak{z}_{1}\right)\right]^{\frac{\Gamma(\gamma+1)}{2(\mathfrak{z}_{2}-\mathfrak{z}_{1})^{\gamma\omega}}}}\right|_{*}\\
&\leq\exp\left\{\frac{\mathfrak{z}_{2}-\mathfrak{z}_{1}}{2}\int_{0}^{\frac{1}{2}}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right|\left(\left(1-\mathfrak{l}\right)\ln\left|\mathscr{V}^{*}\left(\mathfrak{z}_{1}\right)\right|_{*}+\mathfrak{l}\ln\left|\mathscr{V}^{*}\left(\mathfrak{z}_{2}\right)\right|_{*}\right)\mathrm{d}\mathfrak{l}\right\}\\
&\times\exp\left\{\frac{\mathfrak{z}_{2}-\mathfrak{z}_{1}}{2}\int_{\frac{1}{2}}^{1}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|\left(\left(1-\mathfrak{l}\right)\ln\left|\mathscr{V}^{*}\left(\mathfrak{z}_{1}\right)\right|_{*}+\mathfrak{l}\ln\left|\mathscr{V}^{*}\left(\mathfrak{z}_{2}\right)\right|_{*}\right)\mathrm{d}\mathfrak{l}\right\}\\
&\times\exp\left\{\frac{\mathfrak{z}_{2}-\mathfrak{z}_{1}}{2}\int_{0}^{\frac{1}{2}}\left|\frac{2}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|\left(\mathfrak{l}\ln\left|\mathscr{V}^{*}\left(\mathfrak{z}_{1}\right)\right|_{*}+\left(1-\mathfrak{l}\right)\ln\left|\mathscr{V}^{*}\left(\mathfrak{z}_{2}\right)\right|_{*}\right)\mathrm{d}\mathfrak{l}\right\}
\end{aligned}
$$

$$\times \exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\int_{\frac{1}{2}}^{1}\left|\frac{1}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|\left(\mathfrak{l}\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*+(1-\mathfrak{l})\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)d\mathfrak{l}\right\}$$

$$=\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\int_0^{\frac{1}{2}}(1-\mathfrak{l})\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right|d\mathfrak{l}+\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\int_0^{\frac{1}{2}}\mathfrak{l}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right|d\mathfrak{l}\right)\right\}$$

$$\times \exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\int_{\frac{1}{2}}^{1}(1-\mathfrak{l})\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|d\mathfrak{l}+\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\int_{\frac{1}{2}}^{1}\mathfrak{l}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|d\mathfrak{l}\right)\right\}$$

$$\times \exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\int_0^{\frac{1}{2}}\mathfrak{l}\left|\frac{2}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|d\mathfrak{l}+\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\int_0^{\frac{1}{2}}(1-\mathfrak{l})\left|\frac{2}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|d\mathfrak{l}\right)\right\}$$

$$\times \exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\int_{\frac{1}{2}}^{1}\mathfrak{l}\left|\frac{1}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|d\mathfrak{l}+\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\int_{\frac{1}{2}}^{1}(1-\mathfrak{l})\left|\frac{1}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|d\mathfrak{l}\right)\right\}$$

$$=\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*+\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)\left(\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right|d\mathfrak{l}+\int_{\frac{1}{2}}^{1}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|d\mathfrak{l}\right)\right\}$$

$$=\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*+\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)\mathscr{A}_1(\gamma,\omega)+\mathscr{A}_2(\gamma,\omega)\right\}$$

$$=\left(|\mathscr{V}^*(\mathfrak{z}_1)|_*|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\mathscr{A}_1(\gamma,\omega)+\mathscr{A}_2(\gamma,\omega)\right)},$$

where we have used the facts that

$$\mathscr{A}_1(\omega,\gamma)=\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right|d\mathfrak{l} \tag{3.2}$$

$$=\begin{cases}\dfrac{1}{\omega^{\gamma+1}}\left(\omega-\dfrac{4\omega}{3}\left(1-\left(\dfrac{2}{3}\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{\omega}}+B_{1-\left(\frac{1}{2}\right)^{\omega}}\left(1+\gamma,\frac{1}{\omega}\right)-2B_{\left(\frac{2}{3}\right)^{\frac{1}{\gamma}}}\left(1+\gamma,\frac{1}{\omega}\right)\right), & \text{for} \quad \left(\dfrac{2}{3}\right)^{\frac{1}{\gamma}}+\left(\dfrac{1}{2}\right)^{\omega}\leq 1,\\ \dfrac{1}{\omega^{\gamma+1}}\left(\dfrac{\omega}{3}-B_{1-\left(\frac{1}{2}\right)^{\omega}}\left(1+\gamma,\frac{1}{\omega}\right)\right), & \text{for} \quad \left(\dfrac{2}{3}\right)^{\frac{1}{\gamma}}+\left(\dfrac{1}{2}\right)^{\omega}\geq 1,\end{cases}$$

and

$$\mathscr{A}_2(\omega,\gamma)=\int_{\frac{1}{2}}^{1}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|d\mathfrak{l} \tag{3.3}$$

$$=\begin{cases}\dfrac{1}{\omega^{\gamma+1}}\left(B_{\left(\frac{1}{2}\right)^{\omega}}\left(\frac{1}{\omega},1+\gamma\right)-\dfrac{\omega}{6}\right), & \text{for} \quad \left(\dfrac{1}{3}\right)^{\frac{1}{\gamma}}+\left(\dfrac{1}{2}\right)^{\omega}\leq 1,\\ \dfrac{1}{\omega^{\gamma+1}}\left(\dfrac{\omega}{6}-\dfrac{2\omega}{3}\left(1-\left(\dfrac{1}{3}\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{\omega}}-B_{\left(\frac{1}{2}\right)^{\omega}}\left(\frac{1}{\omega},1+\gamma\right)+2B_{1-\left(\frac{1}{3}\right)^{\frac{1}{\gamma}}}\left(\frac{1}{\omega},1+\gamma\right)\right), & \text{for} \quad \left(\dfrac{1}{3}\right)^{\frac{1}{\gamma}}+\left(\dfrac{1}{2}\right)^{\omega}\geq 1.\end{cases}$$

□

Corollary 3.2. *If we attempt to take $\omega=1$, we get the following Milne inequality via Riemann-Liouville fractional multiplicative integrals*

$$\left|\frac{\left[\left(\mathscr{V}(\mathfrak{z}_1)\right)^2\left(\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)^{-1}\left(\mathscr{V}(\mathfrak{z}_2)\right)^2\right]^{\frac{1}{3}}}{\left[\left({}_{\mathfrak{z}_1}\mathscr{J}_*^{\gamma}\mathscr{V}\right)(\mathfrak{z}_2)\left({}_*\mathscr{J}_{\mathfrak{z}_2}^{\gamma}\mathscr{V}\right)(\mathfrak{z}_1)\right]^{\frac{\Gamma(\gamma+1)}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma}}}}\right|_*\leq\left(|\mathscr{V}^*(\mathfrak{z}_1)|_*|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\mathscr{C}_1(\gamma)+\mathscr{C}_2(\gamma)\right)},$$

where $\mathscr{C}_1(\gamma)$ and $\mathscr{C}_2(\gamma)$ are defined as in (1.4) and (1.5), respectively.

Moreover, by setting $\gamma = 1$, we obtain the following Milne inequality for multiplicative integral

$$\left| \frac{\left[(\mathscr{V}(\mathfrak{z}_1))^2 \left(\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)^{-1} (\mathscr{V}(\mathfrak{z}_2))^2 \right]^{\frac{1}{3}}}{\left(\int_{\mathfrak{z}_1}^{\mathfrak{z}_2} (\mathscr{V}(\ell))^{\mathrm{d}\ell} \right)^{\frac{1}{\mathfrak{z}_2-\mathfrak{z}_1}}} \right|_* \leq \left(|\mathscr{V}^*(\mathfrak{z}_1)|_* |\mathscr{V}^*(\mathfrak{z}_2)|_* \right)^{\frac{5(\mathfrak{z}_2-\mathfrak{z}_1)}{24}}.$$

3.2. Milne-Type Inequality for Functions with Bounded *-derivatives

Theorem 3.3. *Let $\mathscr{V} : [\mathfrak{z}_1, \mathfrak{z}_2] \to \mathbb{R}^+$ be a *-differentiable function on $[\mathfrak{z}_1, \mathfrak{z}_2]$, where $0 \leq \mathfrak{z}_1 < \mathfrak{z}_2$. Suppose there exist positive constants $\mathfrak{p}$ and $\mathfrak{q}$ such that*

$$\mathfrak{q} \leq \mathscr{V}^*(\ell) \leq \mathfrak{p}, \quad \text{for all } \ell \in [\mathfrak{z}_1, \mathfrak{z}_2].$$

Then, the following inequality holds:

$$\left| \frac{\left[(\mathscr{V}(\mathfrak{z}_1))^2 \left(\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)^{-1} (\mathscr{V}(\mathfrak{z}_2))^2 \right]^{\frac{1}{3\omega^\gamma}}}{\left[\left({}_{\mathfrak{z}_1}^{\gamma}\mathscr{I}_*^{\omega}\mathscr{V}\right)(\mathfrak{z}_2) \left({}_{*}^{\gamma}\mathscr{I}_{\mathfrak{z}_2}^{\omega}\mathscr{V}\right)(\mathfrak{z}_1) \right]^{\frac{\Gamma(\gamma+1)}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}} \right|_* \leq \left(\frac{\mathfrak{p}}{\mathfrak{q}}\right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}(\mathscr{A}_1(\gamma,\omega)+\mathscr{A}_2(\gamma,\omega))},$$

where $\mathrm{B}(.,.)$ and $\mathrm{B}_\sigma(.,.)$ are the Beta and the incomplete Beta functions, and $\mathscr{A}_1$ and $\mathscr{A}_2$ are defined as in (3.2) and (3.3), respectively.

Proof. Expanding the identity from Lemma 2.1 yields

$$\begin{aligned}
&\frac{\left[(\mathscr{V}(\mathfrak{z}_1))^2 \left(\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)^{-1} (\mathscr{V}(\mathfrak{z}_2))^2 \right]^{\frac{1}{3\omega^\gamma}}}{\left[\left({}_{\mathfrak{z}_1}^{\gamma}\mathscr{I}_*^{\omega}\mathscr{V}\right)(\mathfrak{z}_2) \left({}_{*}^{\gamma}\mathscr{I}_{\mathfrak{z}_2}^{\omega}\mathscr{V}\right)(\mathfrak{z}_1) \right]^{\frac{\Gamma(\gamma+1)}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}} \\
&= \left(\int_0^{\frac{1}{2}} \left((\mathscr{V}^*((1-\mathfrak{l})\mathfrak{z}_1 + \mathfrak{l}\mathfrak{z}_2)) \left(\left(\frac{1-(1-\mathfrak{l})^\omega}{\omega}\right)^\gamma - \frac{2}{3\omega^\gamma} \right) \right)^{\mathrm{d}\mathfrak{l}} \right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}} \left(\int_{\frac{1}{2}}^1 \left((\mathscr{V}^*((1-\mathfrak{l})\mathfrak{z}_1 + \mathfrak{l}\mathfrak{z}_2)) \left(\left(\frac{1-(1-\mathfrak{l})^\omega}{\omega}\right)^\gamma - \frac{1}{3\omega^\gamma} \right) \right)^{\mathrm{d}\mathfrak{l}} \right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}} \\
&\times \left(\int_0^{\frac{1}{2}} \left((\mathscr{V}^*(\mathfrak{l}\mathfrak{z}_1 + (1-\mathfrak{l})\mathfrak{z}_2)) \left(\frac{2}{3\omega^\gamma} - \left(\frac{1-(1-\mathfrak{l})^\omega}{\omega}\right)^\gamma \right) \right)^{\mathrm{d}\mathfrak{l}} \right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}} \left(\int_{\frac{1}{2}}^1 \left((\mathscr{V}^*(\mathfrak{l}\mathfrak{z}_1 + (1-\mathfrak{l})\mathfrak{z}_2)) \left(\frac{1}{3\omega^\gamma} - \left(\frac{1-(1-\mathfrak{l})^\omega}{\omega}\right)^\gamma \right) \right)^{\mathrm{d}\mathfrak{l}} \right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}} \\
&= \exp\left\{ \frac{\mathfrak{z}_2-\mathfrak{z}_1}{2} \int_0^{\frac{1}{2}} \left(\left(\frac{1-(1-\mathfrak{l})^\omega}{\omega}\right)^\gamma - \frac{2}{3\omega^\gamma} \right) \left(\ln(\mathscr{V}^*((1-\mathfrak{l})\mathfrak{z}_1 + \mathfrak{l}\mathfrak{z}_2)) - \frac{\ln(\mathfrak{q})+\ln(\mathfrak{p})}{2} \right) \mathrm{d}\mathfrak{l} \right\} \\
&\times \exp\left\{ \frac{\mathfrak{z}_2-\mathfrak{z}_1}{2} \int_{\frac{1}{2}}^1 \left(\left(\frac{1-(1-\mathfrak{l})^\omega}{\omega}\right)^\gamma - \frac{1}{3\omega^\gamma} \right) \left(\ln(\mathscr{V}^*((1-\mathfrak{l})\mathfrak{z}_1 + \mathfrak{l}\mathfrak{z}_2)) - \frac{\ln(\mathfrak{q})+\ln(\mathfrak{p})}{2} \right) \mathrm{d}\mathfrak{l} \right\} \\
&\times \exp\left\{ \frac{\mathfrak{z}_2-\mathfrak{z}_1}{2} \int_0^{\frac{1}{2}} \left(\frac{2}{3\omega^\gamma} - \left(\frac{1-(1-\mathfrak{l})^\omega}{\omega}\right)^\gamma \right) \left(\ln(\mathscr{V}^*(\mathfrak{l}\mathfrak{z}_1 + (1-\mathfrak{l})\mathfrak{z}_2)) - \frac{\ln(\mathfrak{q})+\ln(\mathfrak{p})}{2} \right) \mathrm{d}\mathfrak{l} \right\} \\
&\times \exp\left\{ \frac{\mathfrak{z}_2-\mathfrak{z}_1}{2} \int_{\frac{1}{2}}^1 \left(\frac{1}{3\omega^\gamma} - \left(\frac{1-(1-\mathfrak{l})^\omega}{\omega}\right)^\gamma \right) \left(\ln(\mathscr{V}^*(\mathfrak{l}\mathfrak{z}_1 + (1-\mathfrak{l})\mathfrak{z}_2)) - \frac{\ln(\mathfrak{q})+\ln(\mathfrak{p})}{2} \right) \mathrm{d}\mathfrak{l} \right\}. \qquad (3.4)
\end{aligned}$$

By applying the *-absolute value on both sides of the inequality (3.4), we get

$$\begin{aligned}
&\left| \frac{\left[(\mathscr{V}(\mathfrak{z}_1))^2 \left(\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)^{-1} (\mathscr{V}(\mathfrak{z}_2))^2 \right]^{\frac{1}{3\omega^\gamma}}}{\left[\left({}_{\mathfrak{z}_1}^{\gamma}\mathscr{I}_*^{\omega}\mathscr{V}\right)(\mathfrak{z}_2) \left({}_{*}^{\gamma}\mathscr{I}_{\mathfrak{z}_2}^{\omega}\mathscr{V}\right)(\mathfrak{z}_1) \right]^{\frac{\Gamma(\gamma+1)}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}} \right|_* \\
&\leq \exp\left\{ \frac{\mathfrak{z}_2-\mathfrak{z}_1}{2} \int_0^{\frac{1}{2}} \left| \left(\frac{1-(1-\mathfrak{l})^\omega}{\omega} \right)^\gamma - \frac{2}{3\omega^\gamma} \right| \left| \ln(\mathscr{V}^*((1-\mathfrak{l})\mathfrak{z}_1 + \mathfrak{l}\mathfrak{z}_2)) - \frac{\ln(\mathfrak{q})+\ln(\mathfrak{p})}{2} \right| \mathrm{d}\mathfrak{l} \right\}
\end{aligned}$$

$$\times \exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\int_{\frac{1}{2}}^{1}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|\left|\ln\left(\mathscr{V}^*\left((1-\mathfrak{l})\mathfrak{z}_1+\mathfrak{l}\mathfrak{z}_2\right)\right)-\frac{\ln(\mathfrak{q})+\ln(\mathfrak{p})}{2}\right|d\mathfrak{l}\right\}$$

$$\times \exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\int_{0}^{\frac{1}{2}}\left|\frac{2}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|\left|\ln\left(\mathscr{V}^*\left(\mathfrak{l}\mathfrak{z}_1+(1-\mathfrak{l})\mathfrak{z}_2\right)\right)-\frac{\ln(\mathfrak{q})+\ln(\mathfrak{p})}{2}\right|d\mathfrak{l}\right\}$$

$$\times \exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\int_{\frac{1}{2}}^{1}\left|\frac{1}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|\left|\ln\left(\mathscr{V}^*\left(\mathfrak{l}\mathfrak{z}_1+(1-\mathfrak{l})\mathfrak{z}_2\right)\right)-\frac{\ln(\mathfrak{q})+\ln(\mathfrak{p})}{2}\right|d\mathfrak{l}\right\}. \tag{3.5}$$

Since the natural logarithm function ln is increasing and $\mathfrak{q}\le\mathscr{V}^*(\ell)\le\mathfrak{p}$, it follows that

$$\ln(\mathfrak{q})\le\ln\left(\mathscr{V}^*(\ell)\right)\le\ln(\mathfrak{p}).$$

As a result, we obtain

$$\left|\ln\left(\mathscr{V}^*(\ell)\right)-\frac{\ln(\mathfrak{q})+\ln(\mathfrak{p})}{2}\right|\le\ln\left(\sqrt{\frac{\mathfrak{p}}{\mathfrak{q}}}\right),\quad \text{for all } \ell\in[\mathfrak{z}_1,\mathfrak{z}_2].$$

Finally, we get from inequality (3.5)

$$\begin{aligned}
&\left|\frac{\left[\left(\mathscr{V}(\mathfrak{z}_1)\right)^2\left(\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)^{-1}\left(\mathscr{V}(\mathfrak{z}_2)\right)^2\right]^{\frac{1}{3\omega^{\gamma}}}}{\left[\left({}^{\gamma}_{\mathfrak{z}_1}\mathscr{J}^{\omega}_{*}\mathscr{V}\right)(\mathfrak{z}_2)\left({}^{\gamma}_{*}\mathscr{J}^{\omega}_{\mathfrak{z}_2}\mathscr{V}\right)(\mathfrak{z}_1)\right]^{\frac{\Gamma(\gamma+1)}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}}\right|_*\\
&\le\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\ln\left(\frac{\mathfrak{p}}{\mathfrak{q}}\right)\left(\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right|d\mathfrak{l}+\int_{\frac{1}{2}}^{1}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|d\mathfrak{l}\right)\right\}\\
&=\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\mathscr{A}_1(\gamma,\omega)+\mathscr{A}_2(\gamma,\omega)\right)\ln\left(\frac{\mathfrak{p}}{\mathfrak{q}}\right)\right\}\\
&=\left(\frac{\mathfrak{p}}{\mathfrak{q}}\right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\mathscr{A}_1(\gamma,\omega)+\mathscr{A}_2(\gamma,\omega)\right)},
\end{aligned}$$

where we have used (3.2) and (3.3). □

3.3. Numerical Example and Graphical Validation

This subsection presents a concrete numerical example accompanied by graphical representations to illustrate the applicability and accuracy of the theoretical results derived earlier. The visualizations highlight how well the proposed bounds align with actual function behavior.

Example 3.4. *Let the function $\mathscr{V}$ be defined on the interval $[\mathfrak{z}_1,\mathfrak{z}_2]=[0,1]$ by $\mathscr{V}(\ell)=\exp\{\ell^2\}$. This function is suitable since its multiplicative derivative $\mathscr{V}^*(\ell)=\exp\{2\ell\}$ is both multiplicatively convex and bounded by 1 and $\exp\{2\}$ on $[0,1]$, satisfying the hypotheses of Theorems 3.1 and 3.3.*

The left term of the inequality provided by Theorem 3.1 and Theorem 3.3 for the considered function is as follows:

$$\begin{aligned}
LHS&=\left|\exp\left\{\frac{7}{12\omega^{\gamma}}-\frac{1}{2\omega^{\gamma}}\left[1+2\gamma\left(B\left(\gamma,\frac{2}{\omega}+1\right)-B\left(\gamma,\frac{1}{\omega}+1\right)\right)\right]\right\}\right|_*\\
&=\exp\left\{\frac{7}{12\omega^{\gamma}}-\frac{1}{2\omega^{\gamma}}\left[1+2\gamma\left(B\left(\gamma,\frac{2}{\omega}+1\right)-B\left(\gamma,\frac{1}{\omega}+1\right)\right)\right]\right\},
\end{aligned}\tag{3.6}$$

and the resultant right term derived from both Theorem 3.1 and Theorem 3.3 is given by

$$RHS=\exp\left\{\mathscr{A}_1(\gamma,\omega)+\mathscr{A}_2(\gamma,\omega)\right\},\tag{3.7}$$

where $\mathscr{A}_1$ and $\mathscr{A}_2$ are defined as in (3.2) and (3.3), respectively.

The left- and right-hand sides of the inequality, given by equations (3.6) and (3.7) respectively, are plotted graphically in Figure 3.1 for $\gamma\in[0.1,4]$ and $\omega\in[0.1,1]$. As can be observed, the left-hand side always lies under the right-hand side, thereby confirming the validity of the result stated in Theorem 3.1 and Theorem 3.3.

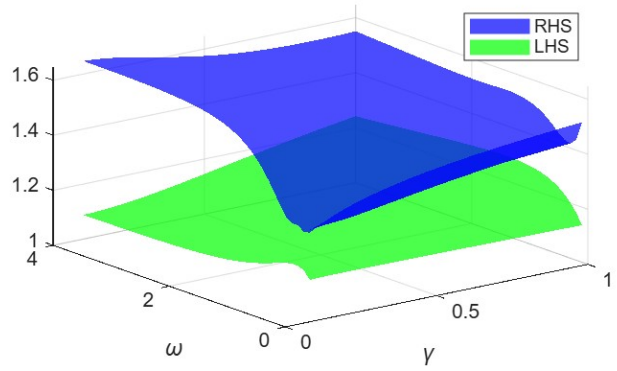

Figure 3.1: Illustration for Theorem 3.1 and Theorem 3.3.

4. Further Results

We extend the analysis to a particular class of functions where the *-derivative satisfies the constraint $(\ln \mathscr{V}^*)^m$, for $m > 1$.

Theorem 4.1. *Let $\mathscr{V} : [\mathfrak{z}_1, \mathfrak{z}_2] \to \mathbb{R}^+$ be a *-differentiable mapping on $[\mathfrak{z}_1, \mathfrak{z}_2]$ with $0 \le \mathfrak{z}_1 < \mathfrak{z}_2$. If $(\ln|\mathscr{V}^*|_*)^m$ is convex on $[\mathfrak{z}_1, \mathfrak{z}_2]$ for $m > 1$, then the following inequality holds*

$$\left| \frac{\left[(\mathscr{V}(\mathfrak{z}_1))^2 \left(\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2} \right) \right)^{-1} (\mathscr{V}(\mathfrak{z}_2))^2 \right]^{\frac{1}{3\omega^\gamma}}}{\left[\left({}^{\gamma}_{\mathfrak{z}_1}\mathscr{I}^{\omega}_{*}\mathscr{V} \right)(\mathfrak{z}_2) \left({}^{\gamma}_{*}\mathscr{I}^{\omega}_{\mathfrak{z}_2}\mathscr{V} \right)(\mathfrak{z}_1) \right]^{\frac{\Gamma(\gamma+1)}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}} \right|_* \le \left(|\mathscr{V}^*(\mathfrak{z}_1)|_* |\mathscr{V}^*(\mathfrak{z}_2)|_* \right)^{(\mathfrak{z}_2-\mathfrak{z}_1)\frac{1}{2}\frac{1+3\frac{1}{m}}{\frac{3+m}{m}}} \left((\mathscr{B}_1(\gamma,\omega))^{\frac{1}{n}} + (\mathscr{B}_2(\gamma,\omega))^{\frac{1}{n}} \right),$$

where $m+n = mn$, and $\mathscr{B}_1$ and $\mathscr{B}_2$ are given by

$$\mathscr{B}_1(\gamma,\omega) = \int_0^{\frac{1}{2}} \left| \left(\frac{1-(1-\mathfrak{l})^\omega}{\omega} \right)^\gamma - \frac{2}{3\omega^\gamma} \right|^n d\mathfrak{l}, \tag{4.1}$$

and

$$\mathscr{B}_2(\gamma,\omega) = \int_{\frac{1}{2}}^1 \left| \left(\frac{1-(1-\mathfrak{l})^\omega}{\omega} \right)^\gamma - \frac{1}{3\omega^\gamma} \right|^n d\mathfrak{l}. \tag{4.2}$$

Proof. By using Hölder inequality along with the convexity of $(\ln|\mathscr{V}^*|_*)^m$, inequality (3.1) yields

$$\left| \frac{\left[(\mathscr{V}(\mathfrak{z}_1))^2 \left(\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2} \right) \right)^{-1} (\mathscr{V}(\mathfrak{z}_2))^2 \right]^{\frac{1}{3\omega^\gamma}}}{\left[\left({}^{\gamma}_{\mathfrak{z}_1}\mathscr{I}^{\omega}_{*}\mathscr{V} \right)(\mathfrak{z}_2) \left({}^{\gamma}_{*}\mathscr{I}^{\omega}_{\mathfrak{z}_2}\mathscr{V} \right)(\mathfrak{z}_1) \right]^{\frac{\Gamma(\gamma+1)}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}} \right|_*$$

$$\le \exp\left\{ \frac{\mathfrak{z}_2-\mathfrak{z}_1}{2} \left(\int_0^{\frac{1}{2}} \left| \left(\frac{1-(1-\mathfrak{l})^\omega}{\omega} \right)^\gamma - \frac{2}{3\omega^\gamma} \right|^n d\mathfrak{l} \right)^{\frac{1}{n}} \left(\int_0^{\frac{1}{2}} (\ln|\mathscr{V}^*((1-\mathfrak{l})\mathfrak{z}_1 + \mathfrak{l}\mathfrak{z}_2)|_*)^m d\mathfrak{l} \right)^{\frac{1}{m}} \right\}$$

$$\times \exp\left\{ \frac{\mathfrak{z}_2-\mathfrak{z}_1}{2} \left(\int_{\frac{1}{2}}^1 \left| \left(\frac{1-(1-\mathfrak{l})^\omega}{\omega} \right)^\gamma - \frac{1}{3\omega^\gamma} \right|^n d\mathfrak{l} \right)^{\frac{1}{n}} \left(\int_{\frac{1}{2}}^1 (\ln|\mathscr{V}^*((1-\mathfrak{l})\mathfrak{z}_1 + \mathfrak{l}\mathfrak{z}_2)|_*)^m d\mathfrak{l} \right)^{\frac{1}{m}} \right\}$$

$$\times \exp\left\{ \frac{\mathfrak{z}_2-\mathfrak{z}_1}{2} \left(\int_0^{\frac{1}{2}} \left| \frac{2}{3\omega^\gamma} - \left(\frac{1-(1-\mathfrak{l})^\omega}{\omega} \right)^\gamma \right|^n d\mathfrak{l} \right)^{\frac{1}{n}} \left(\int_0^{\frac{1}{2}} (\ln|\mathscr{V}^*(\mathfrak{l}\mathfrak{z}_1 + (1-\mathfrak{l})\mathfrak{z}_2)|_*)^m d\mathfrak{l} \right)^{\frac{1}{m}} \right\}$$

$$\times \exp\left\{ \frac{\mathfrak{z}_2-\mathfrak{z}_1}{2} \left(\int_{\frac{1}{2}}^1 \left| \frac{1}{3\omega^\gamma} - \left(\frac{1-(1-\mathfrak{l})^\omega}{\omega} \right)^\gamma \right|^n d\mathfrak{l} \right)^{\frac{1}{n}} \left(\int_{\frac{1}{2}}^1 (\ln|\mathscr{V}^*(\mathfrak{l}\mathfrak{z}_1 + (1-\mathfrak{l})\mathfrak{z}_2)|_*)^m d\mathfrak{l} \right)^{\frac{1}{m}} \right\}$$

$$
\begin{aligned}
&\leq \exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right|^n d\mathfrak{l}\right)^{\frac{1}{n}}\left(\int_0^{\frac{1}{2}}\left[(1-\mathfrak{l})\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\right)^m+\mathfrak{l}\left(\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^m\right]d\mathfrak{l}\right)^{\frac{1}{m}}\right\}\\
&\times\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\int_{\frac{1}{2}}^{1}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|^n d\mathfrak{l}\right)^{\frac{1}{n}}\left(\int_{\frac{1}{2}}^{1}\left[(1-\mathfrak{l})\left(\ln\mathscr{V}^*(\mathfrak{z}_1)\right)^m+\mathfrak{l}\left(\ln\mathscr{V}^*(\mathfrak{z}_2)\right)^m\right]d\mathfrak{l}\right)^{\frac{1}{m}}\right\}\\
&\times\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\int_0^{\frac{1}{2}}\left|\frac{2}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|^n d\mathfrak{l}\right)^{\frac{1}{n}}\left(\int_0^{\frac{1}{2}}\left[\mathfrak{l}\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\right)^m+(1-\mathfrak{l})\left(\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^m\right]d\mathfrak{l}\right)^{\frac{1}{m}}\right\}\\
&\times\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\int_{\frac{1}{2}}^{1}\left|\frac{1}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|^n d\mathfrak{l}\right)^{\frac{1}{n}}\left(\int_{\frac{1}{2}}^{1}\left[\mathfrak{l}\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\right)^m+(1-\mathfrak{l})\left(\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^m\right]d\mathfrak{l}\right)^{\frac{1}{m}}\right\}\\
&=\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right|^n d\mathfrak{l}\right)^{\frac{1}{n}}\left[\left(\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\right)^m\int_0^{\frac{1}{2}}(1-\mathfrak{l})\,d\mathfrak{l}+\left(\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^m\int_0^{\frac{1}{2}}\mathfrak{l}\,d\mathfrak{l}\right)^{\frac{1}{m}}\right.\right.\\
&\left.\left.+\left(\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\right)^m\int_0^{\frac{1}{2}}\mathfrak{l}\,d\mathfrak{l}+\left(\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^m\int_0^{\frac{1}{2}}(1-\mathfrak{l})\,d\mathfrak{l}\right)^{\frac{1}{m}}\right]\right\}\\
&\times\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\int_{\frac{1}{2}}^{1}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|^n d\mathfrak{l}\right)^{\frac{1}{n}}\left[\left(\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\right)^m\int_{\frac{1}{2}}^{1}(1-\mathfrak{l})\,d\mathfrak{l}+\left(\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^m\int_{\frac{1}{2}}^{1}\mathfrak{l}\,d\mathfrak{l}\right)^{\frac{1}{m}}\right.\right.\\
&\left.\left.+\left(\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\right)^m\int_{\frac{1}{2}}^{1}\mathfrak{l}\,d\mathfrak{l}+\left(\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^m\int_{\frac{1}{2}}^{1}(1-\mathfrak{l})\,d\mathfrak{l}\right)^{\frac{1}{m}}\right]\right\}\\
&=\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left[\left(\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right|^n d\mathfrak{l}\right)^{\frac{1}{n}}+\left(\int_{\frac{1}{2}}^{1}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|^n d\mathfrak{l}\right)^{\frac{1}{n}}\right]\right.\\
&\left.\times\left[\left(\frac{3\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\right)^m+\left(\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^m}{8}\right)^{\frac{1}{m}}+\left(\frac{\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\right)^m+3\left(\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^m}{8}\right)^{\frac{1}{m}}\right]\right\}\\
&\leq\exp\left\{(\mathfrak{z}_2-\mathfrak{z}_1)\frac{1+3^{\frac{1}{m}}}{2^{\frac{3+m}{m}}}\left(\left(\mathscr{B}_1(\gamma,\omega)\right)^{\frac{1}{n}}+\left(\mathscr{B}_2(\gamma,\omega)\right)^{\frac{1}{n}}\right)\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*+\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)\right\}\\
&=\left(|\mathscr{V}^*(\mathfrak{z}_1)|_*\,|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^{(\mathfrak{z}_2-\mathfrak{z}_1)\frac{1+3^{\frac{1}{m}}}{2^{\frac{3+m}{m}}}\left(\left(\mathscr{B}_1(\gamma,\omega)\right)^{\frac{1}{n}}+\left(\mathscr{B}_2(\gamma,\omega)\right)^{\frac{1}{n}}\right)},
\end{aligned}
$$

where we have used (4.1), (4.2) and the fact that $\mathscr{N}^m+\mathscr{M}^m\leq(\mathscr{N}+\mathscr{M})^m$ for $\mathscr{N},\mathscr{M}>0$ and $m>1$. □

Theorem 4.2. *Let $\mathscr{V}:[\mathfrak{z}_1,\mathfrak{z}_2]\to\mathbb{R}^+$ be a *-differentiable mapping on $[\mathfrak{z}_1,\mathfrak{z}_2]$ with $0\leq\mathfrak{z}_1<\mathfrak{z}_2$. If $\left(\ln|\mathscr{V}^*|_*\right)^m$ is convex on $[\mathfrak{z}_1,\mathfrak{z}_2]$ for $m>1$, then the following inequality holds*

$$
\left|\frac{\left[\left(\mathscr{V}(\mathfrak{z}_1)\right)^2\left(\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)^{-1}\left(\mathscr{V}(\mathfrak{z}_2)\right)^2\right]^{\frac{1}{3\omega^{\gamma}}}}{\left[\left({}^{\gamma}_{\mathfrak{z}_1}\mathscr{I}^{\omega}_{*}\mathscr{V}\right)(\mathfrak{z}_2)\left({}^{\gamma}_{*}\mathscr{I}^{\omega}_{\mathfrak{z}_2}\mathscr{V}\right)(\mathfrak{z}_1)\right]^{\frac{\Gamma(\gamma+1)}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}}\right|_*
$$

$$\leq \left(|\mathscr{V}^*(\mathfrak{z}_1)|_*|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left[(\mathscr{A}_1(\gamma,\omega))^{1-\frac{1}{m}}\left(\mathscr{D}_1(\gamma,\omega)^{\frac{1}{m}}+\mathscr{D}_2(\gamma,\omega)^{\frac{1}{m}}\right)+(\mathscr{A}_2(\gamma,\omega))^{1-\frac{1}{m}}\left(\mathscr{D}_3(\gamma,\omega)^{\frac{1}{m}}+\mathscr{D}_4(\gamma,\omega)^{\frac{1}{m}}\right)\right]},$$

where $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *are given by (3.2) and (3.3), and* $\mathscr{D}_i(\gamma,\omega)$ *are given by*

$$\mathscr{D}_1(\gamma,\omega)=\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right|(1-\mathfrak{l})\mathrm{d}\mathfrak{l} \tag{4.3}$$

$$=\begin{cases}\frac{1}{\omega^{\gamma+1}}\left[\frac{5\omega}{12}-\frac{2\omega}{3}\left(1-\left(\frac{2}{3}\right)^{\frac{1}{\gamma}}\right)^{\frac{2}{\omega}}+B_{1-\left(\frac{1}{2}\right)^{\omega}}\left(1+\gamma,\frac{2}{\omega}\right)-2B_{\left(\frac{2}{3}\right)^{\frac{1}{\gamma}}}\left(1+\gamma,\frac{2}{\omega}\right)\right] & for \quad \left(\frac{2}{3}\right)^{\frac{1}{\gamma}}+\left(\frac{1}{2}\right)^{\omega}\leq 1\\ \frac{1}{\omega^{\gamma+1}}\left[\frac{\omega}{4}-B_{1-\left(\frac{1}{2}\right)^{\omega}}\left(1+\gamma,\frac{2}{\omega}\right)\right] & for \quad \left(\frac{2}{3}\right)^{\frac{1}{\gamma}}+\left(\frac{1}{2}\right)^{\omega}\geq 1,\end{cases}$$

$$\mathscr{D}_2(\gamma,\omega)=\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right|\mathfrak{l}\mathrm{d}\mathfrak{l}=\mathscr{A}_1(\gamma,\omega)-\mathscr{D}_1(\gamma,\omega), \tag{4.4}$$

$$\mathscr{D}_3(\gamma,\omega)=\int_{\frac{1}{2}}^{1}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|(1-\mathfrak{l})\mathrm{d}\mathfrak{l} \tag{4.5}$$

$$=\begin{cases}\frac{1}{\omega^{\gamma+1}}\left[B\left(1+\gamma,\frac{2}{\omega}\right)-B_{1-\left(\frac{1}{2}\right)^{\omega}}\left(1+\gamma,\frac{2}{\omega}\right)-\frac{\omega}{24}\right] & for \quad \left(\frac{1}{3}\right)^{\frac{1}{\gamma}}+\left(\frac{1}{2}\right)^{\omega}\leq 1\\ \frac{1}{\omega^{\gamma+1}}\left[\frac{\omega}{24}-\frac{\omega}{3}\left(1-\left(\frac{1}{3}\right)^{\frac{1}{\gamma}}\right)^{\frac{2}{\omega}}+B\left(1+\gamma,\frac{2}{\omega}\right)\right.\\ \left.+B_{1-\left(\frac{1}{2}\right)^{\omega}}\left(1+\gamma,\frac{2}{\omega}\right)-2B_{\left(\frac{1}{3}\right)^{\frac{2}{\gamma}}}\left(1+\gamma,\frac{1}{\omega}\right)\right] & for \quad \left(\frac{1}{3}\right)^{\frac{1}{\gamma}}+\left(\frac{1}{2}\right)^{\omega}\geq 1\end{cases}$$

and

$$\mathscr{D}_4(\gamma,\omega)=\int_{\frac{1}{2}}^{1}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|\mathfrak{l}\mathrm{d}\mathfrak{l}=\mathscr{A}_2(\gamma,\omega)-\mathscr{D}_3(\gamma,\omega), \tag{4.6}$$

respectively.

Proof. By using power mean inequality along with the convexity of $(\ln|\mathscr{V}^*|_*)^m$, inequality (3.1) yields

$$\left|\frac{\left[(\mathscr{V}(\mathfrak{z}_1))^2\left(\mathscr{V}\left(\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}\right)\right)^{-1}(\mathscr{V}(\mathfrak{z}_2))^2\right]^{\frac{1}{3\omega^{\gamma}}}}{\left[\left({}_{\mathfrak{z}_1}^{\gamma}\mathscr{J}_*^{\omega}\mathscr{V}\right)(\mathfrak{z}_2)\left({}_{*}^{\gamma}\mathscr{J}_{\mathfrak{z}_2}^{\omega}\mathscr{V}\right)(\mathfrak{z}_1)\right]^{\frac{\Gamma(\gamma+1)}{2(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma\omega}}}}\right|_*$$

$$\leq \exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right|\mathrm{d}\mathfrak{l}\right)^{1-\frac{1}{m}}\left(\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right|(\ln|\mathscr{V}^*((1-\mathfrak{l})\mathfrak{z}_1+\mathfrak{l}\mathfrak{z}_2)|_*)^m\mathrm{d}\mathfrak{l}\right)^{\frac{1}{m}}\right\}$$

$$\times\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\int_{\frac{1}{2}}^{1}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|\mathrm{d}\mathfrak{l}\right)^{1-\frac{1}{m}}\left(\int_{\frac{1}{2}}^{1}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|(\ln|\mathscr{V}^*((1-\mathfrak{l})\mathfrak{z}_1+\mathfrak{l}\mathfrak{z}_2)|_*)^m\mathrm{d}\mathfrak{l}\right)^{\frac{1}{m}}\right\}$$

$$\times\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\int_0^{\frac{1}{2}}\left|\frac{2}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|\mathrm{d}\mathfrak{l}\right)^{1-\frac{1}{m}}\left(\int_0^{\frac{1}{2}}\left|\frac{2}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|(\ln|\mathscr{V}^*(\mathfrak{l}\mathfrak{z}_1+(1-\mathfrak{l})\mathfrak{z}_2)|_*)^m\mathrm{d}\mathfrak{l}\right)^{\frac{1}{m}}\right\}$$

$$\times\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\int_{\frac{1}{2}}^{1}\left|\frac{1}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|\mathrm{d}\mathfrak{l}\right)^{1-\frac{1}{m}}\left(\int_{\frac{1}{2}}^{1}\left|\frac{1}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|(\ln|\mathscr{V}^*(\mathfrak{l}\mathfrak{z}_1+(1-\mathfrak{l})\mathfrak{z}_2)|_*)^m\mathrm{d}\mathfrak{l}\right)^{\frac{1}{m}}\right\}$$

$$
\begin{aligned}
&\leq \exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right|\mathrm{d}\mathfrak{l}\right)^{1-\frac{1}{m}}\left(\int_0^{\frac{1}{2}}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{2}{3\omega^{\gamma}}\right|\left[(1-\mathfrak{l})\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\right)^m+\mathfrak{l}\left(\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^m\right]\mathrm{d}\mathfrak{l}\right)^{\frac{1}{m}}\right\}\\
&\quad\times\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\int_{\frac{1}{2}}^{1}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|\mathrm{d}\mathfrak{l}\right)^{1-\frac{1}{m}}\left(\int_{\frac{1}{2}}^{1}\left|\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}-\frac{1}{3\omega^{\gamma}}\right|\left[(1-\mathfrak{l})\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\right)^m+\mathfrak{l}\left(\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^m\right]\mathrm{d}\mathfrak{l}\right)^{\frac{1}{m}}\right\}\\
&\quad\times\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\int_0^{\frac{1}{2}}\left|\frac{2}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|\mathrm{d}\mathfrak{l}\right)^{1-\frac{1}{m}}\left(\int_0^{\frac{1}{2}}\left|\frac{2}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|\left[\mathfrak{l}\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\right)^m+(1-\mathfrak{l})\left(\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^m\right]\mathrm{d}\mathfrak{l}\right)^{\frac{1}{m}}\right\}\\
&\quad\times\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\int_{\frac{1}{2}}^{1}\left|\frac{1}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|\mathrm{d}\mathfrak{l}\right)^{1-\frac{1}{m}}\left(\int_{\frac{1}{2}}^{1}\left|\frac{1}{3\omega^{\gamma}}-\left(\frac{1-(1-\mathfrak{l})^{\omega}}{\omega}\right)^{\gamma}\right|\left[\mathfrak{l}\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\right)^m+(1-\mathfrak{l})\left(\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^m\right]\mathrm{d}\mathfrak{l}\right)^{\frac{1}{m}}\right\}\\
&=\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\mathscr{D}_1(\gamma,\omega)+\mathscr{D}_2(\gamma,\omega)\right)^{1-\frac{1}{m}}\left(\mathscr{D}_1(\gamma,\omega)\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\right)^m+\mathscr{D}_2(\gamma,\omega)\left(\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^m\right)^{\frac{1}{m}}\right\}\\
&\quad\times\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\mathscr{D}_3(\gamma,\omega)+\mathscr{D}_4(\gamma,\omega)\right)^{1-\frac{1}{m}}\left(\mathscr{D}_3(\gamma,\omega)\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\right)^m+\mathscr{D}_4(\gamma,\omega)\left(\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^m\right)^{\frac{1}{m}}\right\}\\
&\quad\times\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\mathscr{D}_1(\gamma,\omega)+\mathscr{D}_2(\gamma,\omega)\right)^{1-\frac{1}{m}}\left(\mathscr{D}_2(\gamma,\omega)\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\right)^m+\mathscr{D}_1(\gamma,\omega)\left(\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^m\right)^{\frac{1}{m}}\right\}\\
&\quad\times\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left(\mathscr{D}_3(\gamma,\omega)+\mathscr{D}_4(\gamma,\omega)\right)^{1-\frac{1}{m}}\left(\mathscr{D}_4(\gamma,\omega)\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*\right)^m+\mathscr{D}_3(\gamma,\omega)\left(\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^m\right)^{\frac{1}{m}}\right\}\\
&\leq\exp\left\{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left[\left(\mathscr{D}_1(\gamma,\omega)+\mathscr{D}_2(\gamma,\omega)\right)^{1-\frac{1}{m}}\left(\mathscr{D}_1(\gamma,\omega)^{\frac{1}{m}}+\mathscr{D}_2(\gamma,\omega)^{\frac{1}{m}}\right)\right.\right.\\
&\quad\left.\left.+\left(\mathscr{D}_3(\gamma,\omega)+\mathscr{D}_4(\gamma,\omega)\right)^{1-\frac{1}{m}}\left(\mathscr{D}_3(\gamma,\omega)^{\frac{1}{m}}+\mathscr{D}_4(\gamma,\omega)^{\frac{1}{m}}\right)\right]\left(\ln|\mathscr{V}^*(\mathfrak{z}_1)|_*+\ln|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)\right\}\\
&=\left(|\mathscr{V}^*(\mathfrak{z}_1)|_*|\mathscr{V}^*(\mathfrak{z}_2)|_*\right)^{\frac{\mathfrak{z}_2-\mathfrak{z}_1}{2}\left[\left(\mathscr{D}_1(\gamma,\omega)+\mathscr{D}_2(\gamma,\omega)\right)^{1-\frac{1}{m}}\left(\mathscr{D}_1(\gamma,\omega)^{\frac{1}{m}}+\mathscr{D}_2(\gamma,\omega)^{\frac{1}{m}}\right)+\left(\mathscr{D}_3(\gamma,\omega)+\mathscr{D}_4(\gamma,\omega)\right)^{1-\frac{1}{m}}\left(\mathscr{D}_3(\gamma,\omega)^{\frac{1}{m}}+\mathscr{D}_4(\gamma,\omega)^{\frac{1}{m}}\right)\right]},
\end{aligned}
$$

where we have used (4.3)-(4.6), and the fact that $\mathscr{N}^m+\mathscr{M}^m\leq(\mathscr{N}+\mathscr{M})^m$ for $\mathscr{N},\mathscr{M}>0$ and $m>1$. □

5. Application to Special Means

To demonstrate the utility of our findings beyond theoretical interest, we apply the established inequalities to special means such as arithmetic, geometric, and logarithmic means. These applications provide insight into how multiplicative fractional tools can be used in practical mathematical problems.

Proposition 5.1. *Let $\mathfrak{z}_1,\mathfrak{z}_2\in\mathbb{R}$ with $0<\mathfrak{z}_1<\mathfrak{z}_2$, then we have*

$$
\left|\frac{\exp\left\{\frac{4\mathscr{A}\left(\mathfrak{z}_1^2+\mathfrak{z}_2^2\right)-\mathscr{A}^2(\mathfrak{z}_1,\mathfrak{z}_2)}{3\gamma\omega}\right\}}{\exp\left\{\left(\frac{1}{\omega^{\gamma}}\mathscr{A}\left(\mathfrak{z}_1^2,\mathfrak{z}_2^2\right)+\frac{\left(\mathfrak{z}_2-\mathfrak{z}_1\right)^2}{\omega^{\gamma+1}}\left(2B\left(\frac{2}{\omega},\gamma+1\right)-B\left(\frac{1}{\omega},\gamma+1\right)\right)\right)\right\}}\right|_*\leq\exp\left\{\left(\mathfrak{z}_2^2-\mathfrak{z}_1^2\right)\left(\mathscr{A}_1(\gamma,\omega)+\mathscr{A}_2(\gamma,\omega)\right)\right\},
$$

where $\mathscr{A}(.,.)$ represents the arithmetic mean expressed as $\mathscr{A}(\mathfrak{z}_1,\mathfrak{z}_2)=\frac{\mathfrak{z}_1+\mathfrak{z}_2}{2}$, and $\mathscr{A}_1$ and $\mathscr{A}_2$ are defined as in (3.2) and (3.3).

Proof. To prove the claim, we apply Theorem 3.1 to the particular case in which the function is defined as $\mathscr{V}(\ell)=\exp\left\{\ell^2\right\}$. Under this setting, the associated *-derivative is given by $\mathscr{V}^*(\ell)=\exp\{2\ell\}$, and the corresponding conformable fractional multiplicative integrals are expressed as

$$
{}^{\gamma}_{\mathfrak{z}_1}\mathscr{I}^{\omega}_{*}\mathscr{V}(\mathfrak{z}_2)=\exp\left\{\frac{\left(\mathfrak{z}_2-\mathfrak{z}_2\right)^{\gamma\omega}}{\Gamma(\gamma+1)\omega^{\gamma+1}}\left(\omega\mathfrak{z}_1^2+2\left(\mathfrak{z}_2-\mathfrak{z}_1\right)\mathfrak{z}_1B\left(\frac{1}{\omega},\gamma+1\right)+2\left(\mathfrak{z}_2-\mathfrak{z}_1\right)^2B\left(\frac{2}{\omega},\gamma+1\right)\right)\right\},
$$

and

$$
{}^{\gamma}_{*}\mathscr{I}^{\omega}_{\mathfrak{z}_2}\mathscr{V}(\mathfrak{z}_1)=\exp\left\{\frac{\left(\mathfrak{z}_2-\mathfrak{z}_1\right)^{\gamma\omega}}{\omega^{\gamma+1}\Gamma(\gamma+1)}\left(\omega\mathfrak{z}_2^2-2\left(\mathfrak{z}_2-\mathfrak{z}_1\right)\mathfrak{z}_2B\left(\frac{1}{\omega},\gamma+1\right)+2\left(\mathfrak{z}_2-\mathfrak{z}_1\right)^2B\left(\frac{2}{\omega},\gamma+1\right)\right)\right\}.
$$

□

Proposition 5.2. *Let $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{R}$ with $0 < \mathfrak{z}_1 < \mathfrak{z}_2$ and $\delta > 1$, then we have*

$$\left| \exp\left\{ \left(\mathscr{A}\left(\frac{\gamma}{\gamma+\delta-1}, (\delta+\gamma) B(\gamma+1, \delta) \right) + \frac{2^{\delta}-1}{3\gamma \times 2^{\delta-1}} \right) (\mathfrak{z}_2 - \mathfrak{z}_1)^{\delta-1} \right\} \right|_* \leq \exp\left\{ (\delta-1)(\mathfrak{z}_2-\mathfrak{z}_1)^{\delta-1} (\mathscr{A}_1(1,\gamma) + \mathscr{A}_2(1,\gamma)) \right\}.$$

Proof. To establish the claim, we apply the first inequality of corollary 3.2 to the specific case where the function is defined as $\mathscr{V}(\ell) = \exp\left\{(\ell-\mathfrak{z}_1)^{\delta-1}\right\}$. In this setting, the corresponding *derivative becomes $\mathscr{V}^*(\ell) = \exp\left\{(\delta-1)(\ell-\mathfrak{z}_1)^{\delta-2}\right\}$ and the corresponding Riemann-Liouville fractional multiplicative integrals are given by

$${}_{\mathfrak{z}_1}\mathscr{J}_*^{\gamma}\mathscr{V}(\mathfrak{z}_2) = \exp\left\{ \frac{\Gamma(\delta)}{\Gamma(\delta+\gamma)} (\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma+\delta-1} \right\}$$

and

$${}_*\mathscr{J}_{\mathfrak{z}_2}^{\gamma}\mathscr{V}(\mathfrak{z}_1) = \exp\left\{ \frac{(\mathfrak{z}_2-\mathfrak{z}_1)^{\gamma+\delta-1}}{(\gamma+\delta-1)\Gamma(\gamma)} \right\}.$$

□

6. Final Remarks and Future Directions

In this work, we have explored Milne's inequality through the lens of conformable fractional multiplicative integrals, extending classical results into the non-Newtonian framework. By introducing a new multiplicative identity, we derived Milne-type inequalities for functions whose *-derivatives exhibit multiplicative convexity as well as functions with bounded *-derivatives. Our approach not only generalizes known results but also provides tighter bounds in certain settings. The numerical example, along with its graphical representation, illustrates the validity and practical relevance of the proposed framework. Furthermore, we extended the analysis to functions satisfying additional derivative conditions and demonstrated applications to special means. These findings contribute to the growing field of multiplicative fractional calculus and open new perspectives for future investigations, particularly in the development of advanced integral inequalities and their computational applications.

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