Alexandria Engineering Journal (2020) 59, 4709–4717

HOSTED BY

Alexandria University

Alexandria Engineering Journal

www.elsevier.com/locate/aej
www.sciencedirect.com

Numerical solution of fractional diffusion equation by Chebyshev collocation method and residual power series method

Mine Aylin Bayrak [a,*], Ali Demir [a], Ebru Ozbilge [b]

[a] Department of Mathematics, Kocaeli University, Izmit, Kocaeli, Turkey
[b] Department of Mathematics and Statistics, American University of the Middle East, Egaila, Kuwait

Received 31 May 2020; revised 21 August 2020; accepted 24 August 2020
Available online 19 September 2020

KEYWORDS

Chebyshev collocation method;
Fractional diffusion equation;
Caputo fractional derivatives;
Residual power series method

Abstract In this paper, we propose an efficient Chebyshev collocation scheme to solve diffusion problem including time fractional diffusion equation considering the fractional derivative in the Liouville-Caputo sense. By making use of shifted Chebyshev polynomial series and their orthogonality properties, the problem is reduced to the system of fractional ordinary differential equations which can be solved by residual power series method (RPSM) with the help of the given scheme and boundary conditions. The numerical examples shows that the method is reliable and effective to construct the numerical solution of fractional diffusion equation.

© 2020 The Authors. Published by Elsevier B.V. on behalf of Faculty of Engineering, Alexandria University. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

1. Introduction

Recently, the fractional calculus attracts the growing interest of many scientists since it is an important tool to model various systems in sciences and engineering. Using fractional derivatives leads to more degree of freedom which makes mathematical models including fractional derivatives more realistic than ones including integer order derivatives. Therefore, it has many applications in many areas of science such as image processing, earthquake and biomedical engineering, viscoelasticity, finance, hydrology, signal processing, biology and control system [1–9,33].

However, it is much more harder to obtain analytical solution of fractional diffusion equations (FDEs). As a result for the solutions of many FDEs the numerical techniques such as Adomian decomposition method [10], Variational iteration method [11], Taylor collocation method [12,13], Homotopy analysis method [14,25,31], Collection method [15,16], Gegenbauer Wavelet Method [26] and other numerical and spectral methods [17,21,27,28,32,34–40,42] etc. are employed.

The main goal in this article is to study the numerical solution of the following time fractional diffusion problem:

$$D_t^{\alpha}u(x,t) = \kappa(x)u_{xx}(x,t) + F(x,t), \tag{1}$$

$$u(x,0) = \varphi(x), \tag{2}$$

$$u(0,t) = \mu_1(t), \tag{3}$$

$$u(1,t) = \mu_2(t), \tag{4}$$

where $0 < \alpha \leqslant 1, 0 \leqslant x,t \leqslant 1, \kappa(x)$ is the diffusion coefficient and $F(x,t)$ is the source function. This problem with integer order derivative have been studied by many scientists [29,30].

* Corresponding author.
E-mail address: aylin@kocaeli.edu.tr (M.A. Bayrak).

https://doi.org/10.1016/j.aej.2020.08.033
1110-0168 © 2020 The Authors. Published by Elsevier B.V. on behalf of Faculty of Engineering, Alexandria University.
This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

The motivation of this study is to construct the numerical solution of problem (1)–(4) by utilizing the combination of Chebyshev collocation method and RPSM. Taking the orthogonality properties of shifted Chebyshev polynomials allows us to have a system of FDEs which can be solved by RPSM [43,44].

We take the following steps to form the approximate solution of problem (1)–(4): Chebyshev collocation method is used to construct the spatial part of the solution, including space variable x, which comes from the integer-order part of the problem. Later, RPSM is utilized to form the part of the solution, including time variable t, which comes from the fractional order part of the problem. This provides a sufficient novelty since Chebyshev collocation methods are suitable for integer-order problems and RPSM is suitable for fractional-order problems.

This paper is organized as follows. In Section 2, mathematical preliminaries used in this paper, are given. In Section 3, Chebyshev polynomials and their features are given. In Section 4, the algorithm of the proposed method for solving time fractional diffusion problem is presented. In Section 5, the reliability and accuracy of the proposed method are demonstrated by illustrative examples. Finally, the concluded results are presented in Section 6.

2. Preliminaries

In this section the fundamentals for fractional calculus theory are given [2,18–20].

Definition 1. The Riemann–Liouville fractional integral of order $\alpha(\alpha \geqslant 0)$ is given as [2]

$$J^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt, \alpha > 0, x > 0,$$

$$J^0 f(x) = f(x).$$

Definition 2. The Liouville-Caputo fractional derivative of order α is given as [2]

$$D^{\alpha}f(x) = J^{m-\alpha}D^m f(x) = \int_0^x (x-t)^{m-\alpha-1}\frac{d^m}{dt^m}f(t)dt, m-1 < \alpha < m, x > 0,$$

where D^m is the classic differential operator with order m.

Definition 3. The α^{th} order Liouville-Caputo time fractional derivative of $u(x,t)$ is defined as [2]

$$D_t^{\alpha}u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\xi)^{m-\alpha-1}\frac{\partial^m u(x,\xi)}{\partial t^m}d\xi, & m-1 < \alpha < m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m \in N. \end{cases}$$

Definition 4. A power series expansion of the form

$$\sum_{m=0}^{\infty} c_m(t-t_0)^{m\alpha} = c_0 + c_1(t-t_0)^{\alpha} + c_2(t-t_0)^{2\alpha} + \ldots,$$
$$0 \leqslant m-1 < \alpha \leqslant m, t \geqslant t_0$$

is called fractional power series about $t = t_0$ [2].

Definition 5. The power series expansions about $t = t_0$

$$\sum_{k=0}^{\infty}\sum_{l=0}^{n-1} f_k(t-t_0)^{k\alpha+l}, 0 \leqslant n-1 < \alpha \leqslant n, t_0 \leqslant t < t_0 + R$$

is called multiple fractional power series about $t = t_0$ [2].

3. Chebyshev polynomials

Chebyshev polynomials are used in various mathematical problems such as for approximate solutions of ODEs and FDEs. Because of orthogonality properties of them, they are used in various areas of numerical analysis as least square approximation, uniform approximation. By making use of Chebyshev polynomials, a uniform and accurate approximation of a function $f(x)$ defined on $[a,b]$, can be constructed by a polynomial $P(x)$. Moreover, the functions in the basis of the FDEs are approximated by Chebyshev polynomials [22–24,41]. It is well-known that the first Chebyshev polynomials are defined on the interval $[-1,1]$ as follows [23]:

$$T_n(t) = \cos(n\theta), n = 0,1,2,\ldots, \tag{5}$$

where $t = \cos(\theta)$. The following recursive relations leads to Chebyshev polynomials $T_n(t), n = 1,2,\ldots,$

$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t), T_0(t) = 1, T_1(t) = t. \tag{6}$$

The orthogonality condition is followed from

$$\int_{-1}^{1}\frac{T_i(t)T_j(t)}{\sqrt{1-t^2}}dx = \begin{cases} 0, & i \neq j, \\ \frac{\pi}{2}, & i = j \neq 0, \\ \pi, & i = j = 0. \end{cases} \tag{7}$$

The analytic form of the polynomials $T_n(t)$ can be given in the following form

$$T_n(t) = n\sum_{i=0}^{\left[\frac{n}{2}\right]}(-1)^i 2^{n-2i-1}\frac{(n-i-1)!}{(i)!(n-2i)!}t^{n-2i}, \tag{8}$$

where $\left[\frac{n}{2}\right]$ gives the greatest integer value of $\frac{n}{2}$.

It is very common to use the interval $[0,1]$ instead of the interval $[-1,1]$. Therefore, we need shifted Chebyshev polynomials, obtained by defining the change of variable $t = 2x-1$. The polynomials $\widehat{T}_n(x)$ can be obtained from recurrence relation

$$\widehat{T}_n(x) = T_n(2x-1) = T_{2n}\left(\sqrt{x}\right), \tag{9}$$

where

$$\widehat{T}_{n+1}(x) = (4x-2)\widehat{T}_n(x) - \widehat{T}_{n-1}(x), \widehat{T}_0(x) = 1, \widehat{T}_1(x) = 2x-1.$$

The orthogonality properties of the shifted Chebyshev polynomials are followed by the relation below

$$\int_0^1 \frac{T_i(x)T_j(x)}{\sqrt{x-x^2}}dx = \delta_{i,j}h_j, \tag{10}$$

where $h_k = \frac{\pi}{2}b_k$ with $b_0 = 1, b_k = 1, k \geqslant 1$.

The shifted Chebyshev polynomials $\widehat{T}_n(x)$ can be obtained in the series form as follows

$$\widehat{T}_n(x) = n\sum_{i=0}^{n}(-1)^{n-i}2^{2i}\frac{(n+i-1)!}{(2i)!(n-i)!}x^i, \tag{11}$$

where $\widehat{T}_n(0) = (-1)^n$, $\widehat{T}_n(1) = 1$. For the numerical solution of the FDEs with boundary conditions, the following series in terms of shifted Chebyshev polynomials is used:

$$u(x,t) = \sum_{i=0}^{\infty} c_i(t)\,\widehat{T}_i(x), \tag{12}$$

where $c_i(t), i = 0,1,2,\ldots$.are unknown coefficients. The first m unknown functions $c_i(t), i = 0,1,2,\ldots,m$ are determined as follows

$$c_i(t) = \begin{cases} \frac{1}{\pi}\int_0^1 \frac{u(x,t)\widehat{T}_0(x)}{\sqrt{x-x^2}}dx, & i = 0, \\ \frac{2}{\pi}\int_0^1 \frac{u(x,t)\widehat{T}_i(x)}{\sqrt{x-x^2}}dx, & i = 1,2,\ldots,m, \end{cases} \tag{13}$$

which leads to the following m^{th} degree approximation in the following form

$$u_m(x,t) = \sum_{i=0}^{m} c_i(t)\,\widehat{T}_i(x). \tag{14}$$

Theorem 2. *The α^{th} order fractional derivative of the polynomial approximation $u(x,t)$ which is defined in (14) is computed as follows:*

$$D_x^{\alpha}u_m(x,t) = \sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} c_i(t)\Omega_{i,k}^{(\alpha)}x^{k-\alpha}, \tag{15}$$

where the weight function $\Omega_{i,k}^{(\alpha)}$ is defined as

$$\Omega_{i,k}^{(\alpha)} = (-1)^{i-k}\frac{2^{2k}i(i+k-1)!\Gamma(k+1)}{(i-k)!(2k)!\Gamma(k+1-\alpha)}.$$

4. Algorithm of the proposed method

In order to construct the approximate solution $u(x,t)$ for the problem (1)–(4) by Chebyshev collocation method, the following steps are taken:

Step 1. Plugging the Eq. (14) and Eq. (15) into the Eq. (1) leads to the following equation:

$$\sum_{i=0}^{m} D_t^{\alpha}c_i(t)\,\widehat{T}_i(x) = \kappa(x)\sum_{i=\lceil 2\rceil}^{m}\sum_{k=\lceil 2\rceil}^{i} c_i(t)\Omega_{i,k}^{(2)}x^{k-2} + F(x,t). \tag{16}$$

Step 2. Collocating Eq. (16) at the roots $x_k, k = 0,1,\ldots m-\lceil\alpha\rceil$of the shifted Chebyshev polynomial $\widehat{T}_{m+1-\lceil\alpha\rceil}(x)$, we have a system of fractional ordinary differential equations:

$$\sum_{i=0}^{m} D_t^{\alpha}c_i(t)\,\widehat{T}_i(x_k) = \kappa(x_r)\sum_{i=\lceil 2\rceil}^{m}\sum_{k=\lceil 2\rceil}^{i} c_i(t)\Omega_{i,k}^{(2)}x_r^{k-2} + F(x_r,t), \tag{17}$$

$x_r, r = 0,1,2,\ldots,m-\lceil\alpha\rceil, m-1 < \alpha \leqslant m.$

Step 3. Plugging the Eq. (14) into in the initial and boundary conditions Eq. (2)–(4) leads to the following a system of algebraic equations, we can obtain $(\lceil\alpha\rceil+1)$ equations as follows:

$$\sum_{i=0}^{m} c_i(0)\,\widehat{T}_i(x_r) = \varphi(x_r), \sum_{i=0}^{m} c_i(t)(-1)^i = \mu_1(t), \sum_{i=0}^{m} c_i(t) = \mu_2(t), \tag{18}$$

$x_r, r = 0,1,2,\ldots,m-\lceil\alpha\rceil, m-1 < \alpha \leqslant m.$

Step 4. As a result, we have a system including fractional ordinary differential equations and algebraic equations. Solving this system by RPSM yields unknown functions $c_i(t), i = 0,1,2\ldots m$ which are taken into account to form the approximate solution $u_m(x,t)$.

5. Numerical tests

In this section, time fractional diffusion equations are solved numerically by combination of shifted Chebyshev polynomials and RPSM with the help of MATLAB.

Example 1. Let us consider the following problem

$$D_t^{\alpha}u(x,t) = \frac{1}{2}x^2u_{xx}(x,t), \tag{19}$$

$$u(x,0) = x^2, \tag{20}$$

$$u(0,t) = 0, \tag{21}$$

$$u(1,t) = E_{\alpha}(t^{\alpha}), \tag{22}$$

which has the analytic solution $u(x,t) = x^2E_{\alpha}(t^{\alpha})$. By following the steps of the proposed method the approximate solution for $m = 2$ are obtained in the following way:

$$u_2(x,t) = \sum_{i=0}^{2} c_i(t)\,\widehat{T}_i(x). \tag{23}$$

Step 1. Using Eq. (16), we have

$$\sum_{i=0}^{2} D_t^{\alpha}c_i(t)\,\widehat{T}_i(x) = \frac{1}{2}\sum_{i=\lceil 2\rceil}^{2}\sum_{k=\lceil 2\rceil}^{i} c_i(t)\Omega_{i,k}^{(2)}x^k, \tag{24}$$

Step 2. Using Eq. (17), we have

$$\sum_{i=0}^{2} D_t^{\alpha}c_i(t)\,\widehat{T}_i(x_0) = \frac{1}{2}\sum_{i=\lceil 2\rceil}^{2}\sum_{k=\lceil 2\rceil}^{i} c_i(t)\Omega_{i,k}^{(2)}x_0^k. \tag{25}$$

Hence the explicit form of Eq. (25) become

$$D_t^{\alpha}c_0(t) - D_t^{\alpha}c_2(t) = 2c_2(t) \tag{26}$$

where the root of the shifted Chebyshev polynomial $\widehat{T}_1(x), x_0 = 0.5$ is the collocation point.

Step 3. By taking initial and boundary conditions into account, we have

$$\sum_{i=0}^{2} c_i(0)\,\widehat{T}_i(x_0) = x_0^2, \sum_{i=0}^{2} c_i(t)(-1)^i = 0, \sum_{i=0}^{2} c_i(t) = E_{\alpha}(t^{\alpha}), \tag{27}$$

Hence the explicit form of Eqs. (27) become

$$c_0(0) - c_2(0) = 0.25, \tag{28}$$

$$c_0(t) - c_1(t) + c_2(t) = 0, \tag{29}$$

$$c_0(t) + c_1(t) + c_2(t) = E_{\alpha}(t^{\alpha}). \tag{30}$$

Step 4. RPSM is utilized to determine the first unknown functions $c_0(t)$ is determined from the following fractional differential equation:

$$D_t^{\alpha} c_0(t) + c_0(t) = 0.75E_{\alpha}(t^{\alpha}) \tag{31}$$

based on to the initial condition

$$c_0(0) = 0.375 = f_0. \tag{32}$$

In order to determine the coefficient f_1 in Eq. (31), the truncated series $c_{00}(0) = f_0 + f_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}$ is plugged into the 1-st truncated residual functions to obtain

$$Res_{00} = f_1 + f_0 + f_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)} - 0.75E_{\alpha}(t^{\alpha}). \tag{33}$$

Substituting $t = 0$ in Eq. (33) leads to the following

$$f_1 = 0.375. \tag{34}$$

In a similar way, the unknown coefficients $f_2, f_3, \ldots$ are computed. Hence unknown functions $c_0(t)$ is obtained as follows:

$$c_0(t) = 0.375\left(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}\right), \tag{35}$$

Finally, the unknown functions $c_1(t)$ and $c_2(t)$ are determined from algebraic Eqs. (29) and (30) as follows:

$$c_1(t) = \frac{E_{\alpha}(t^{\alpha})}{2}, \tag{36}$$

$$c_2(t) = \frac{E_{\alpha}(t^{\alpha})}{2} - c_0(t). \tag{37}$$

For $m = 2, 4^{th}$ degree of approximate solution $u_2(x,t)$ is constructed as follows:

$$u_2(x,t) = c_0(t) + c_1(t)(2x-1) + c_2(t)\left(8x^2 - 8x + 1\right). \tag{38}$$

It is clear from the Fig. 1 that the components $c_i(t), i = 0, 1, 2$ of the approximate solution gets closer to ones of analytic solution for $\alpha = 1$ as the number of terms in RPSM increases to 4.

As it can be seen from Table 1 the absolute error of the approximate solution increases until mid-point of x and then it decreases which means that there is a symmetry with respect to mid-point of space variable. Moreover, the approximate solution gets closer to the exact solution as fractional order α increases to 1 which is also obvious from Fig. 2. As it is clear from Table 1 and Fig. 3 and Fig. 4 that the comparison of this method with HAM support our claim that in this way one of the best approximated solution is obtained. Fig. 5, Fig. 6 and Fig. 7 represent the $3D$ space–time graph comparison of exact and approximate solutions when $\alpha = 0.9$.

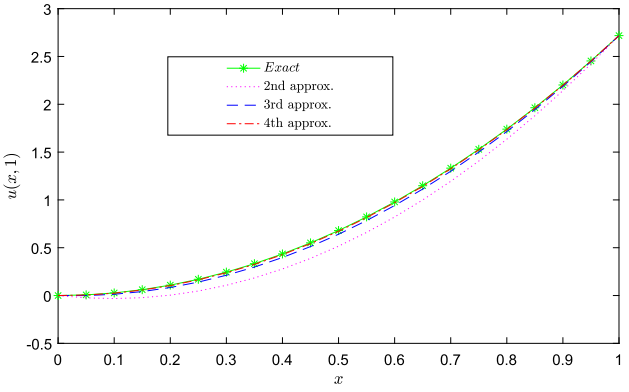


Fig. 1 The graphics of approximate solutions for Ex. 1 at $m = 2$ when $\alpha = 1$.

Example 2. Let us consider the following problem

$$D_t^{\alpha} u(x,t) = -\frac{1}{2}x^2 u_{xx}(x,t) + \left(x^2 - x^3\right)t + \left(x^2 - 3x^3\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \tag{39}$$

$$u(x,0) = 0, \tag{40}$$

$$u(0,t) = u(1,t) = 0, \tag{41}$$

which has the analytic solution $u(x,t) = x^2(1-x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$. By following the steps of the proposed method, the approximate solution for $m = 3$ are obtained in the following form:

$$u_3(x,t) = \sum_{i=0}^{3} c_i(t)\,\widehat{T}_i(x). \tag{42}$$

Using Eq. (17), we have

$$\sum_{i=0}^{3} D_t^{\alpha} c_i(t)\,\widehat{T}_i(x_r) = -\frac{1}{2}\sum_{i=[2]}^{3}\sum_{k=[2]}^{i} c_i(t)\Omega_{i,k}^{(2)} x_r^k + F(x_r,t), \tag{43}$$

where

$$F(x_r,t) = \left(x_r^2 - x_r^3\right)t + \left(x_r^2 - 3x_r^3\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, r = 0, 1.$$

The explicit form of Eq. (43) can be rewritten as follows:

$$\begin{aligned} &D_t^{\alpha} c_0(t) + 0.7072 D_t^{\alpha} c_1(t) + 0.0003 D_t^{\alpha} c_2(t) - 0.7068 D_t^{\alpha} c_3(t) \\ &= -5.8291 c_2(t) - 24.7339 c_3(t) + 0.1067 t - 1.1373 \tfrac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \end{aligned} \tag{44}$$

$$\begin{aligned} &D_t^{\alpha} c_0(t) - 0.7072 D_t^{\alpha} c_1(t) + 0.0003 D_t^{\alpha} c_2(t) + 0.7068 D_t^{\alpha} c_3(t) \\ &= -0.1715 c_2(t) + 0.7276 c_3(t) + 0.0183 t + 0.0120 \tfrac{t^{\alpha+1}}{\Gamma(\alpha+2)}. \end{aligned} \tag{45}$$

By taking initial and boundary conditions into account, we have the following algebraic equations:

$$\sum_{i=0}^{3} c_i(0)\,\widehat{T}_i(x_r) = 0, \sum_{i=0}^{3} c_i(t)(-1)^i = 0, \sum_{i=0}^{3} c_i(t) = 0. \tag{46}$$

In explicit form they can be written as follows:

$$c_0(0) + 0.7072 c_1(0) + 0.0003 c_2(0) - 0.7068 c_3(0) = 0, \tag{47}$$

$$c_0(0) - 0.7072 c_1(0) + 0.0003 c_2(0) + 0.7068 c_3(0) = 0, \tag{48}$$

$$c_0(t) - c_1(t) + c_2(t) - c_3(t) = 0, \tag{49}$$

$$c_0(t) + c_1(t) + c_2(t) + c_3(t) = 0. \tag{50}$$

The collocation points are taken as the root of the shifted Chebyshev polynomial $\widehat{T}_2(x)$ are $x_0 = 0.8536, x_1 = 0.1464$. RPSM is utilized to determine the first-four unknown functions $c_i(t), i = 0, 1, 2, 3$ are determined as follows:

$$D_t^{\alpha} c_0(t) - 3.0012 c_0(t) = 12.0068 c_1(t) + 0.0625 t - 1.1252 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \tag{51}$$

$$D_t^{\alpha} c_1(t) - 9.0033 c_1(t) = 2.0006 c_0(t) + 0.0313 t - 0.4064 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \tag{52}$$

based on to the initial conditions

$$c_0(0) = 0 = f_0, c_1(0) = 0 = g_0. \tag{53}$$

Table 1 Comparison of absolute errors for our method at $m = 2$ and HAM solution ($\hbar = -1$) with $t = 1$ of Example 1 for various values of α.

	$\alpha = 1$		$\alpha = 0.9$		$\alpha = 0.7$	
x	Our method	HAM	Our method	HAM	Our method	HAM
0	0	0	0	0	0	0
0.1	2.69e−03	5.16e−04	6.53e−03	9.89e−04	3.52e−02	3.43e−03
0.2	4.78e−03	2.06e−03	1.16e−02	3.96e−03	6.25e−02	1.37e−02
0.3	6.27e−03	4.65e−03	1.52e−02	8.90e−03	8.20e−02	3.09e−02
0.4	7.16e−03	8.26e−03	1.74e−02	1.58e−03	9.38e−02	5.49e−02
0.5	7.46e−03	1.29e−02	1.81e−02	2.47e−02	9.77e−02	8.58e−02
0.6	7.16e−03	1.86e−02	1.74e−02	3.56e−02	9.38e−02	1.24e−01
0.7	6.27e−03	2.53e−02	1.52e−02	4.85e−02	8.20e−02	1.68e−01
0.8	4.78e−03	3.30e−02	1.16e−02	6.33e−02	6.25e−02	2.20e−01
0.9	2.69e−03	4.18e−02	6.53e−03	8.01e−02	3.52e−02	2.78e−01
1	0	5.16e−02	0	9.89e−02	0	3.43e−01
CPU Time [s]	0.125	0.125	0.125	0.125	0.140625	0.171875

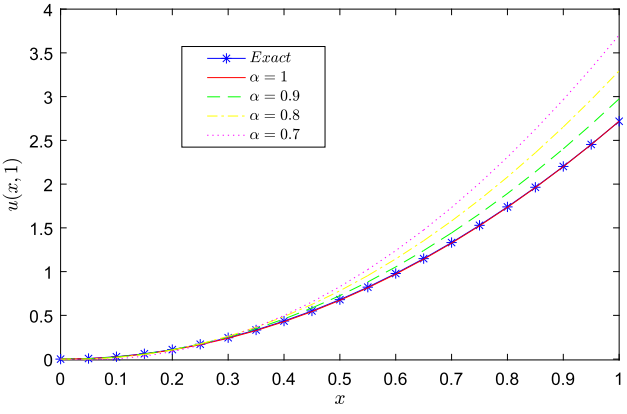


Fig. 2 The graphics of approximate solutions for Ex. 1 at $m = 2$ for various values of fractional order α.

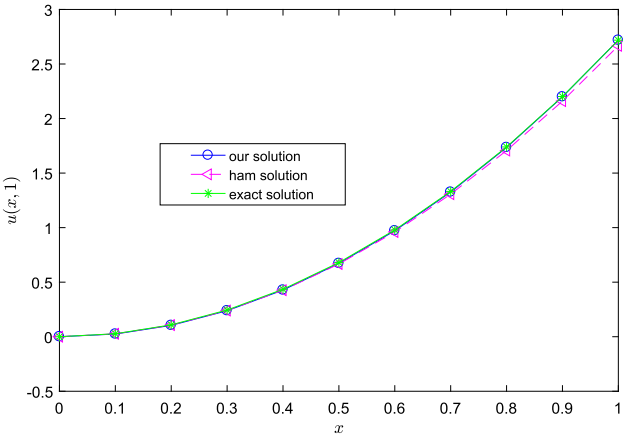


Fig. 3 Comparison of approximation solutions of Ex. 1 with exact solution when $\alpha = 1$ and $t = 1$.

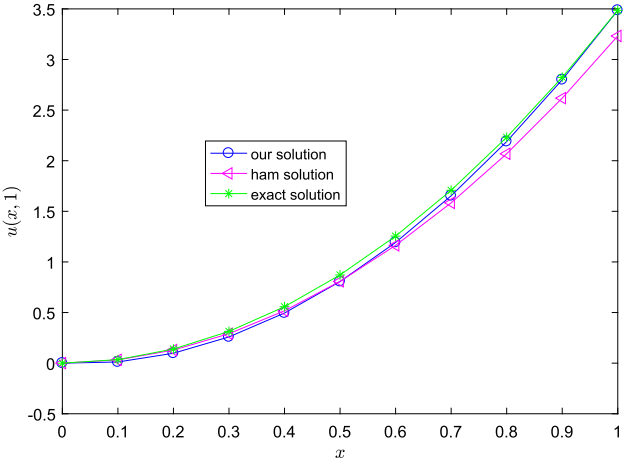


Fig. 4 Comparison of approximation solutions of Ex. 1 with exact solution when $\alpha = 0.75$ and $t = 1$.

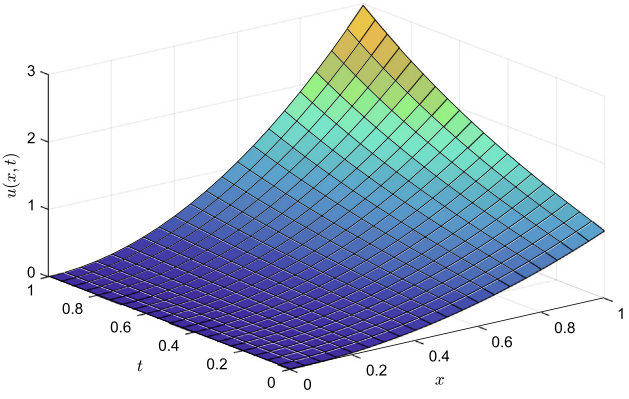


Fig. 5 The graphics of exact solution for Ex. 1 with $\alpha = 0.9$.

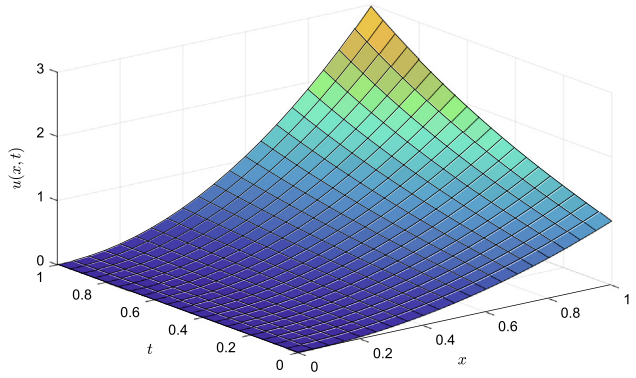


Fig. 6 The graphics of our solution for Ex. 1 with $\alpha = 0.9$.

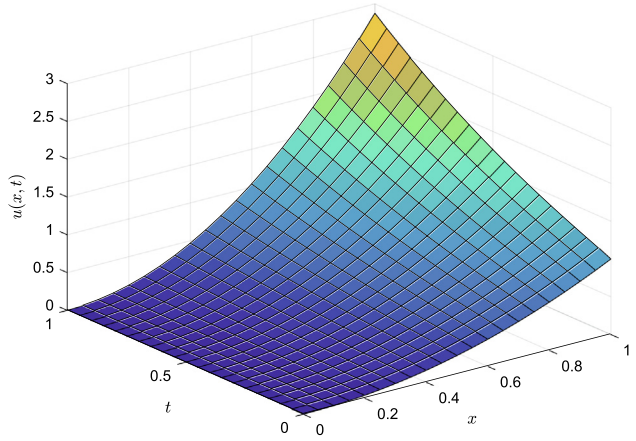


Fig. 7 The graphics of HAM solution for Ex. 1 with $\alpha = 0.9$.

In order to determine the coefficients f_1 and g_1 in Eq. (51), the truncated series $c_{00}(0) = f_0 + f_1 \frac{t^\alpha}{\Gamma(1+\alpha)}$ and $c_{10}(0) = g_0 + g_1 \frac{t^\alpha}{\Gamma(1+\alpha)}$ are plugged into the 1-st truncated residual functions to obtain

$$Res_{00} = f_1 - 3.0012\left(f_0 + f_1 \frac{t^\alpha}{\Gamma(1+\alpha)}\right) - 12.0068\left(g_0 + g_1 \frac{t^\alpha}{\Gamma(1+\alpha)}\right) 0.0625t + 0.5628 \times \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \tag{54}$$

$$Res_{10} = g_1 - 2.0006\left(f_0 + f_1 \frac{t^\alpha}{\Gamma(1+\alpha)}\right) - 9.0033\left(g_0 + g_1 \frac{t^\alpha}{\Gamma(1+\alpha)}\right) - 0.0313t + 0.4064 \times \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}. \tag{55}$$

Substituting $t = 0$ in Eqs. (54)and (55) leads to the following

$$f_1 = 0, g_1 = 0. \tag{56}$$

In a similar way, the unknown coefficients f_2 and g_2 are computed as follows:

$$f_2 = 0.0625, g_2 = 0.0313. \tag{57}$$

The recurrence relation among the coefficients the solution of problem is constructed by repeating this procedure as follows:

$$c_0(t) = 0.0625 \frac{t^{\alpha+1}}{\Gamma(2+\alpha)}, \tag{58}$$

$$c_1(t) = 0.0313 \frac{t^{\alpha+1}}{\Gamma(2+\alpha)}, \tag{59}$$

$$c_2(t) = -c_0(t), \tag{60}$$

$$c_3(t) = -c_1(t). \tag{61}$$

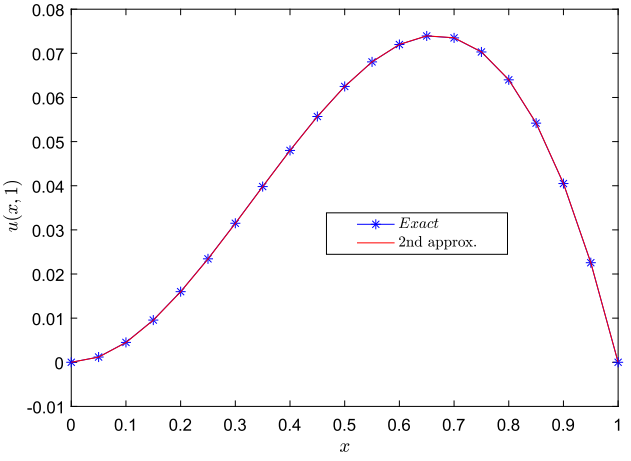


Fig. 8 The graphics of approximate solutions for Ex. 2 at $m = 3$ when $\alpha = 1$.

Table 2 Comparison of absolute errors for our method at $m = 3$ and HAM solution ($\hbar = -1$) with $t = 1$ of Example 2 for various values of α.

	$\alpha = 1$		$\alpha = 0.9$		$\alpha = 0.7$	
x	Our method	HAM	Our method	HAM	Our method	HAM
0	0	0	0	0	0	0
0.1	2.88e−05	1.17e−03	3.15e−05	1.49e−03	3.73e−05	2.35e−03
0.2	3.84e−05	2.67e−03	4.20e−05	3.41e−03	4.97e−05	5.37e−03
0.3	3.36e−05	1.50e−03	3.68e−05	1.92e−03	4.35e−05	3.02e−03
0.4	1.92e−05	5.33e−03	2.10e−05	6.82e−03	2.49e−05	1.07e−02
0.5	0	2.08e−02	0	2.66e−02	0	4.19e−02
0.6	1.92e−05	4.80e−02	2.10e−05	6.14e−02	2.49e−05	9.66e−02
0.7	3.36e−05	8.98e−02	3.68e−05	1.15e−01	4.35e−05	1.81e−01
0.8	3.84e−05	1.49e−01	4.20e−05	1.91e−01	4.97e−05	3.01e−01
0.9	2.88e−05	2.30e−01	3.15e−05	2.93e−01	3.73e−05	3.52 + 00
1	0	3.33e−01	0	4.26e−01	0	6.71e−01
CPU Time [s]	0.015625	0.203125	0.03125	0.203125	0.03125	0.203125

For $m = 3, 2^{th}$ degree approximate solution $u_3(x,t)$ is constructed as follows:

$$u_3(x,t) = c_0(t) + c_1(t)(2x-1) + c_2(t)\left(8x^2 - 8x + 1\right) + c_3(t)\left(32x^3 - 48x^2 + 18x - 1\right). \quad (62)$$

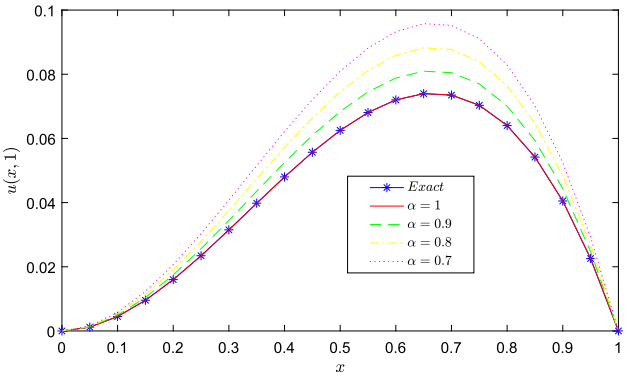


Fig. 9 The graphics of approximate solutions for Ex. 2 at $m = 3$ for various values of fractional order α.

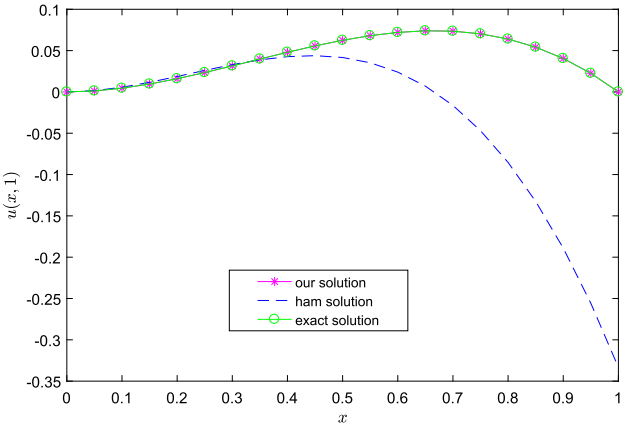


Fig. 10 Comparison of approximation solutions of Ex. 2 with exact solution when $\alpha = 1$ and $t = 1$.

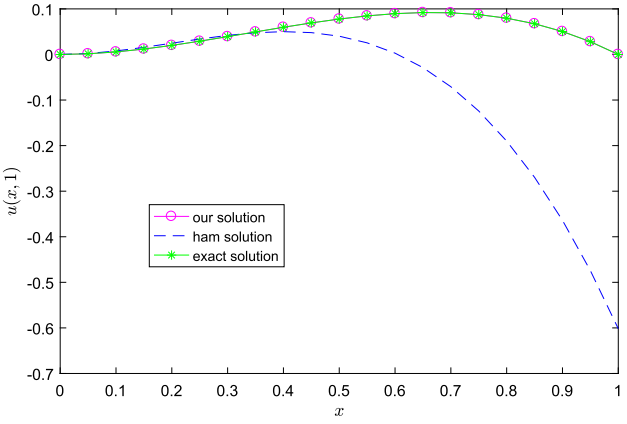


Fig. 11 Comparison of approximation solutions of Ex. 2 with exact solution when $\alpha = 0.75$ and $t = 1$.

It is clear from the Fig. 5 that the components $c_i(t), i = 0, 1, 2, 3$ of the approximate solution gets closer to ones of analytic solution for $\alpha = 1$ as the number of terms in RPSM increases to 2.

As it can be seen from Table 2, the absolute error has the symmetry with respect to mid-point of the space variable x. Moreover, the approximate solution gets closer to the exact solution as fractional order α increases to 1 which is also obvious from Fig. 8. As it is clear from Table 2, Fig. 9 and Fig. 10 that the comparison of this method with HAM support our claim that in this way, one of the best approximated solution

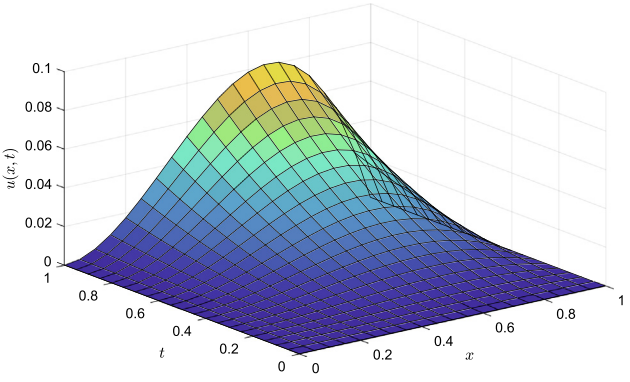


Fig. 12 The graphics of exact solution for Ex. 2 with $\alpha = 0.9$.

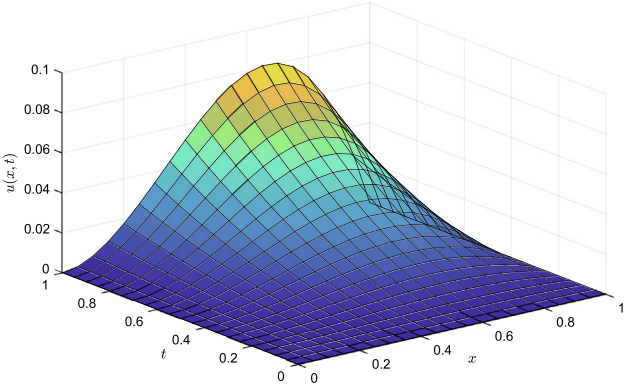


Fig. 13 The graphics of our solution for Ex. 2 with $\alpha = 0.9$.

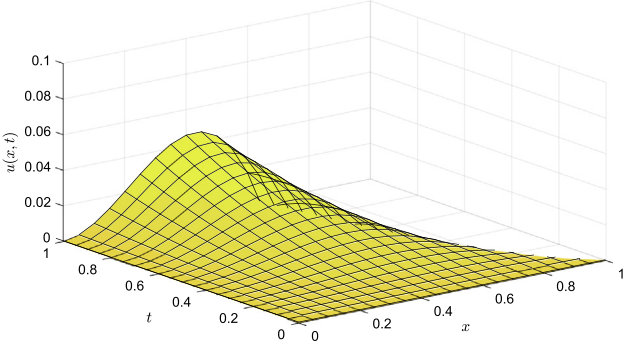


Fig. 14 The graphics of HAM solution for Ex. 1 with $\alpha = 0.9$.

is obtained. Fig. 11, Fig. 12 and Fig. 13 represent the $3D$ space–time graph comparison of exact and approximate solutions when $\alpha = 0.9$ (see Fig. 14).

6. Conclusion

The numerical solution of time fractional diffusion problem is constructed by the combination of Chebyshev collocation method and RPSM. First, by employing Chebyshev collocation method, time fractional diffusion equation is reduced into a system of fractional ordinary differential equations. Secondly, using initial and boundary conditions, we obtain a system of algebraic equations. As a result, we have a system involving fractional ordinary equations and algebraic equations. Later, RPSM is taken into account to solve this system for unknown coefficients of the approximate solution. Finally, utilizing the unknown functions, we construct the approximate solution of the problem.

The results show that the obtained approximation is one of the best since Chebyshev polynomials have orthogonality conditions and are very related to the Fourier and Laurent series. Moreover, the best choice for the degree of the approximation is determined by the highest power of space variable x included in the analytic solution of fractional diffusion equation for integer order. As a result, Chebyshev collocation scheme plays a central role in the construction of approximate solutions of fractional diffusion equation.

In the future research, we apply this method or modification of this method to various problem in science.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors are grateful to the referees for their comments and suggestions which have improved the paper. This research is not funded by any organization.

References

[1] R.L. Bagley, P.J. Torvik, Fractional calculus – a different approach to the analysis of viscoelastically damped structures, AIAA J. 21 (5) (1983) 741–748.
[2] I. Podlubny, Fractional differential equations, Mathematics in Science and Engineering, vol. 198, Academic Press, 1999.
[3] R.P. Agarwal, M. Belmekki, M. Benchohra, A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative, Adv. Differ. Equ. 2009 (2009) 981728.
[4] K.S. Hedrih, The transversal creeping vibrations of a fractional derivative order constitutive relation of nonhomogeneous beam, Math. Probl. Eng. 2006 (5) (2006) 61–78.
[5] P.J. Torvik, R.L. Bagley, On the appearance of the fractional derivative in the behaviour of real materials, J. Appl. Mech. ASME 51 (2) (1984) 294–298.
[6] S.S. Ray, R.K. Bera, Analytical solution of the Bagley-Torvik equation by Adomian decomposition method, Appl. Math. Comput. 168 (1) (2005) 398–410.
[7] S. Manabe, A suggestion of fractional-order controller for flexible spacecraft attitude control, Nonlinear Dyn. 29 (1) (2002) 251–268.
[8] S. Das, Solution of extraordinary differential equations with physical reasoning by obtaining modal reaction series, Model. Simul. Eng. 2010 (2010) 1–19. Article ID 739675.
[9] Y. Zheng, Z. Zhao, The time discontinuous space-time finite element method for fractional diffusion-wave equation, Appl. Num. Math. 150 (2020) 105–116.
[10] Y. Hu, Y. Luo, Z. Lu, Analytical solution of the linear fractional differential equation by Adomian decomposition method, J. Comput. Appl. Math. 215 (1) (2008) 220–229.
[11] A. Ghorbani, A. Alavi, Application of He's variational iteration method to solve semi differential equations of nth order, Math. Probl. Eng. 627983 (2008) 1–10.
[12] J. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Meth. Appl. Mech. Eng. 167 (1–2) (1998) 57–68.
[13] M. Seifollahi, A.S. Shamloo, Numerical solution of nonlinear multi-order fractional differential equations by operational matrix of chebyshev polynomials, World Appl. Prog. 3 (3) (2013) 85–92.
[14] A.A. Hemeda, Homotopy perturbation method for solving partial differential equations of fractional order, Int. J. Math. Anal. 6 (49) (2012) 2431–2448.
[15] M.M. Khadar, N.H. Sweilam, A.M.S. Mahdy, An efficient numerical method for solving the fractional diffusion equation, J. Appl. Math. Bioinformatics 1 (2) (2011) 1–12.
[16] M.M. Khader, On the numerical solutions for the fractional diffusion equation, Commun. Nonlin. Sci. Num. Simul. (2010), https://doi.org/10.1016/j.cnsns.2010.09.007.
[17] E.H. Doha, A.H. Bhrawy, D. Baleanu, S.S. Ezz-Eldien, The operational matrix formulation of the Jacobi Tau approximation for space fractional diffusion equation, Adv. Differ. Equ. 2014 (2014) 231.
[18] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Vol 204, Elsevier Science, 2006.
[19] J.T. Machado, V. Kiryakova, F. Mainardi, Recent history of fractional calculus, Commun. Nonlin. Sci. Num. Simulat. 16 (2011) 1140–1153.
[20] Y. Tian, A. Chen, The existence of positive solution to three-point singular boundary value problem of fractional differential equation, Abstr. Appl. Anal. 2009 (2009) 314656.
[21] R. Magin, X. Feng, D. Baleanu, Solving fractional order Bloch equation, Concept Magn. Res. 34A (1) (2009) 16–23.
[22] J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer, New York, USA, 1996.
[23] J.C. Mason, D.C. Handscomb, Chebyshev Polynomials, Chapman and Hall, New York, 2003.
[24] P. Agarwal, A.A. El-Sayed, Non-standard finite difference and Chebyshev collocation methods for solving fractional diffusion equation, Physica A 500 (2018) 40–49.
[25] H.M. Srivastava, R.M. Jena, S. Chakraverty, S.K. Jena, Dynamic response analysis of fractionally-damped generalized Bagley-Torvik equation subject to external loads, Russian J. Math. Phys. 27 (2020) 254–268.
[26] H.M. Srivastava, F.A. Shah, R. Abass, An application of the Gegenbauer Wavelet method for the numerical solution of the fractional Bagley-Torvik equation, Russian J. Math. Phys. 26 (2019) 77–93.
[27] H. Singh, H.M. Srivastava, Jacobi collocation method for the approximate solution of some fractional-order Riccati differential equations with variable coefficients, Physica A 523 (2019) 1130–1149.
[28] D. Baleanu, B. Shiri, H.M. Srivastava, M. Al-Qurashi, A Chebyshev spectral method based on operational matrix for

fractional differential equations involving non-singular Mittag-Leffler kernel, Adv. Differ. Equ. 2018 (353) (2018) 1–23.
[29] K.V. Zhukovsky, H.M. Srivastava, Analytical solutions for heat diffusion beyond Fourier law, Appl. Math. Comput. 293 (2017) 423–437.
[30] F. Gao, X. Yang, H.M. Srivastava, Exact traveling-wave solutions for linear and non-linear heat transfer equations, Therm. Sci. 21 (6A) (2017) 2307–2311.
[31] K.M. Saad, E.H.F. Al-Shareef, A.K. Alomari, D. Baleanu, J.F. Gómez-Aguilar, On exact solutions for time-fractional Korteweg-de Vries and Korteweg-de Vries-Burger's equations using homotopy analysis transform method, Chin. J. Phys. 63 (2020) 149–162.
[32] M.M. Khader, K.M. Saad, Numerical studies of the fractional Korteweg-de Vries-Burgers' and Burgers' equations, Proceed. Nat. Acad. Sci. India Sec. A: Phys. Sci. (2020).
[33] K.M. Saad, H.M. Srivastava, J.F. Gómez-Aguilar, A fractional Quadratic autocatalysis associated with chemical clock reactions involving linear inhibition, Chaos Solit. Fract. 132 (2020) 109557.
[34] M.M. Khader, K.M. Saad, Numerical treatment for studying the blood ethanol concentration systems with different forms of fractional derivatives, Intern. J. Modern Phys. C 31 (03) (2020) 2050044-1–2050044-13.
[35] K.M. Saad, New fractional derivative with non-singular kernel for deriving Legendre-spectral collocation method, Alexandria Eng. J. 59 (2020) 1909–1917.
[36] H. Singh, H.M. Srivastava, Numerical simulation fractional-order Bloch equation arising in nuclear magnetic resonance by using the Jacobi polynomials, Appl. Sci. 10 (8) (2020) 2850-1–2850-18.
[37] H. Singh, R.K. Pandey, H.M. Srivastava, Solving non-linear fractional variational problems using Jacobi polynomials, Math. 7 (3) (2019) 224-1–224-24.
[38] H. Singh, H.M. Srivastava, D. Kumar, A reliable algorithm for the approximate solution of the nonlinear Lane-Emden type equations arising in astrophysics, Num. Meth. Part. Diff. Eq. 34 (5) (2018) 1524–1555.
[39] H. Singh, H.M. Srivatava, D. Kumar, A reliable numerical algorithm for the fractional vibration equation, Chaos Solit. Fract. 103 (2017) 131–138.
[40] H. Singh, M.R. Sahoo, O.P. Singh, Numerical method based on Galerkin approximation for the fractional Advection-Dispersion equation, Int. J. Appl. Comput. Math. 3 (2017) 2171–2187.
[41] H. Singh, C.S. Singh, A reliable method based on second kind Chebyshev polynomial for the fractional model of Bloch equation, Alexandria Eng. J. 57 (2018) 1425–1432.
[42] H. Singh, F.A. Ghassabzadeh, E. Tohidi, C. Cattani, Legendre spectral method for the fractional Bratu problem, Math. Meth. Appl. Sci. 43 (9) (2020) 5941–5952.
[43] M.A. Bayrak, A. Demir, A new approach for space-time fractional partial differential equations by residual power series method, Appl. Comput. Math. 336 (2018) 215–230.
[44] A. Demir, M.A. Bayrak, E. Ozbilge, New approaches for the solution of space-time fractional Schrödinger equation, Adv. Diff. Eq. 133 (2020) 1–21.