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ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A FIRST AND SECOND-ORDER DISCRETE BOUNDARY VALUE PROBLEM

ARZU ERDEM COŞKUN, ALI DEMIR

Abstract. To purpose of the first part of this paper is to study the nonlinear first order discrete boundary value problem subjected to two-point boundary condition. We establish a new sufficient condition for the existence and uniqueness of the nonlinear first order difference evolution equation. The existence and uniqueness of solutions are proven by means of Banach contraction principle. We define an iteration procedure and apply our results to show that iterated solutions converge to the solutions of the discrete problem. We prove the iteration procedure is terminated when the residual is acceptably small. To illustrate the existence of a unique solution of the contraction mapping theorem some examples are included. In the second part of this study, we consider the following second-order difference boundary value problem

$$\frac{\Delta^2 x_i}{\tau^2} = f\left(t_i, x_i\right), i = 0, 1, 2, ..., n-1,$$
$$ux_0 + vx_n = 0,$$
$$\overline{u}\frac{\Delta x_0}{\tau} + \overline{v}\frac{\Delta x_n}{\tau} = 0,$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, and the difference operator is defined as follows

$$\Delta^2 x_i = \begin{cases} x_{i+1} - 2x_i + x_{i-1}, \text{ for } i = 1, 2, \ldots, n-2, \\ 0, \text{ for } i = 0 \text{ or } i = n-1, \end{cases}.$$

By using Banach fixed point theorem we establish some sufficient conditions for the existence and uniqueness of the above second-order difference boundary value problem. To illustrate the existence of a unique solution of the contraction mapping theorem an example is included.

1. Introduction

The rapid development of the theory of boundary value problems is due to the modeling and simulation of certain nonlinear problems such as the bidomain model for cardiac tissue, ionic conductance in motor nerves of vertebrates, tissue filtration, gas exchange in lungs, calcium dynamics, crystal growth, adsorbate layers on a crystal surface, ionic conductors, glassy materials and others enforced in a natural

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manner. In many cases, a boundary value problem can be constructed by using an appropriate discrete boundary value problem and vice versa. Therefore in recent years, we witness that the discrete boundary value problems serve as mathematical models in diverse fields, such as economy, biology, physics, mechanics, computer science, finance. Several approaches such as fixed point theorem, critical point theory and global bifurcation techniques have been proposed for solving for the discrete boundary value problems. In the literature, Tisdell [12] has presented that under sufficient conditions, potential solutions to the equations will satisfy certain a priori bounds; and the equations will admit at least one solution. by using fixed-point operator theory. In [13], Tisdell has investigated a strategic choice of metric and utilization of Hölder's inequality. The existence of solutions has been proven by the contraction mapping theorem and the Brouwer fixed point theorem in Euclidean space by Mohamed [8]. Ma, Gao and Xu have proved the existence of positive periodic solutions of the nonlinear discrete boundary value problem based upon the global bifurcation techniques. A transform method for solving initial-boundary-value problems for linear semidiscrete (differential-difference) and fully discrete (difference-difference) evolution equations has been presented by Biondini and Wang [2].

In the first part of this work we consider the nonlinear first order discrete boundary value problem

$$\frac{\Delta x_i}{\tau} = f\left(t_i, x_i\right), i = 0, 1, 2, ..., n-1 \tag{1.1}$$

$$ux_0 + vx_n = w, \tag{1.2}$$

which arises as a discrete approximation to the continuous boundary value problem subjected to two-point boundary condition

$$x'\left(t\right) = f\left(t, x\right), t \in [0, T]$$
$$ux\left(0\right) + vx\left(T\right) = w,$$

Here $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function and $\tau = T/n$ is the time step size, grid points (t_i, x_i) are defined as $t_i = i\tau,\ x_i = x\left(t_i\right),\ i = 0, 1, 2, ..., n,\ \Delta x_i = x_{i+1} - x_i$ is referred to as a forward difference and u, v, w are constants satisfying $u + v \neq 0$.

By the solution to (1.1) we understand a vector $x = (x_0, x_2, ..., x_n) \in \mathbb{R}^{n+1}$ such that the equation in (1.1) holds for every $i = 0, 1, 2, ..., n$ and two-point boundary condition (1.2) is satisfied as well. First of all we transform the problem (1.1)−(1.2) into its operator form, [13]

$$x = Sx + \frac{w}{u+v}. \tag{1.3}$$

In this case, $S : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is nonlinear operator defined by

$$\left(Sx\right)_i = h\sum_{j=0}^{n-1} G\left(i, j\right) f\left(t_j, x_j\right), i = 0, 1, 2, ..., n. \tag{1.4}$$

with

$$G\left(i, j\right) = \begin{cases} \frac{u}{u+v}, 0 \leq j \leq i-1 \\ -\frac{v}{u+v}, i \leq j \leq n-1 \end{cases} \tag{1.5}$$

Further we introduce the norm $\|.\|_2$, denoted by

$$\|x\|_2 = \left(\sum_{i=0}^{n} |x_i|^2 \right)^{1/2}$$

and the function $g(t,x) = f(t,x) - f(t,0)$ satisfying

$$|g(t,x)| \leq L|x| \tag{1.6}$$

Under the assumption that the function f satisfies a Lipschitz condition of the form

$$|f(t,x) - f(t,y)| \leq L|x-y|$$

our first task is to establish a new sufficient condition then use Banach fixed point theorem to prove the existence and uniqueness of solutions to $(1.1)-(1.2)$. On the other hand, we use an iteration procedure so that numerical computation of the unique solution is possible. We show that the iteration converges under the same sufficient condition. After that we describe termination criterion. Finally, these results are accompanied by numerous examples.

In the second part of this paper, we discuss the existence of a unique solution for the second order difference equation with two-point boundary conditions

$$\frac{\Delta^2 x_i}{\tau^2} = f(t_i, x_i), i = 0,1,2,...,n-1, \tag{1.7}$$

$$ux_0 + vx_n = 0, \tag{1.8}$$

$$\overline{u}\frac{\Delta x_0}{\tau} + \overline{v}\frac{\Delta x_n}{\tau} = 0, \tag{1.9}$$

corresponding to the discrete version of the following second-order evolution equation:

$$x''(t) = f(t,x), t \in [0,T],$$
$$ux(0) + vx(T) = 0,$$
$$\overline{u}x'(0) + \overline{v}x'(T) = 0,$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies a Lipschitz condition of the form

$$|f(t,x) - f(t,y)| \leq L|x-y|,$$

$\tau = T/n$ is the time step size, grid points (t_i, x_i) are defined as $t_i = i\tau$, $x_i = x(t_i)$, $i = 0,1,2,...,n$, the second difference is given by

$$\Delta^2 x_i = \begin{cases} x_{i+1} - 2x_i + x_{i-1}, \text{ for } i = 1,2,\ldots,n-2, \\ 0, \text{ for } i = 0 \text{ or } i = n-1, \end{cases}$$

and $\Delta x_0 = x_1 - x_0$, $\Delta x_n = x_n - x_{n-1}$, $u, v, \overline{u}, \overline{v}$ are constants with $u^2 + v^2 \neq 0$, $\overline{u}^2 + \overline{v}^2 \neq 0$.

The second-order difference equations have been studied extensively by many authors for existence, asymptotic behavior and continuous dependence to initial values. Among them, Yanqiong Lu and Ruyun Ma [6] have established sufficient conditions for the existence of positive periodic solutions of the second-order difference equation

$$\Delta^2 u(t-1) = \frac{g(t)}{u^{\mu}(t)} - \frac{h(t)}{u^{\lambda}(t)} + f(t).$$

In [7], they have also studied the existence of positive periodic solutions of the second-order difference equation

$$\Delta^2 u(t-1) + a(t)\, u(t-1) = f(t, u(t)) + c(t),\ t \in \mathbb{Z},$$

via Schauder's fixed point theorem, where f is T-periodic with respect to t. Thanin Sitthiwirattham and Jessada Tariboon [10] have been interested in the existence of positive solutions of the following second-order difference equation with three-point summation boundary value problem

$$\Delta^2 u(t-1) + a(t) f(u(t)) = 0, t \in \{1, 2, 3, \ldots, T\},$$
$$u(0) = \beta \sum_{s=1}^{\eta} u(s),\ u(T+1) = \alpha \sum_{s=1}^{\eta} u(s).$$

Jianye Xia and Yuji Liu [14] have established sufficient conditions for the existence of at least one solution of the generalized Neumann boundary value problems for second order nonlinear difference equations

$$\begin{aligned} \nabla\Delta x(k) &= f(k, x(k), x(k+1)),\ k = 1, 2, ..., n-1, \\ x(0) &= ax(1), x(n) = bx(n-1). \end{aligned}$$

D.R. Anderson, I. Rachunková, C.C. Tisdell [1] have investigated the following discrete Neumann boundary value problem

$$\begin{aligned} \nabla\Delta y(k) &= f(k, y(k), y(k+1)),\ k = 1, 2, ..., n-1, \\ \Delta y(0) &= \Delta y(n) = 0. \end{aligned}$$

Tieshan He, Yimin Lu, Youfa Lei, and Fengjian Yang [3] have concerned with the existence of nontrivial periodic solutions and positive periodic solutions for a nonlinear second-order difference equation

$$\begin{aligned} \Delta^2 u(t-1) + q(t) u(t) &= f(t, u(t)),\ t \in \mathbb{Z}, \\ u(t) &= u(t+T). \end{aligned}$$

Fenghua Tan and Zhiming Guo [11] have obtained for the existence and multiplicity of periodic solutions to the following non-linear second-order difference equation

$$\Delta^2 x_{n-1} + f(x_n) = 0,$$

by using the critical point theory. Ignacio Bajo and Eduardo Liz [4] have described the asymptotic behavior and the stability properties of the solutions to the nonlinear second-order difference equation

$$x_{n+1} = \frac{x_{n-1}}{a + b x_n x_{n-1}}, n \geq 0$$

with any initial condition $(x_{-1}, x_0) \in \mathbb{R}^2$. Behzad Djafari Rouhani, Hadi Khatibzadeh [9] have investigated the asymptotic behavior of solutions to the following system of second order nonhomogeneous difference equation

$$u_{n+1} - (1 + \theta_n) u_n + \theta_n u_{n-1} \in c_n A u_n + f_n, n \geq 1$$
$$u_0 = x,\ \sup_{n \geq 0} |u_n| < \infty$$

Yuming Shi and Huaqing Sun [15] have considered self-adjoint extensions for the following second-order symmetric linear difference equation

$$-\nabla.(p(t)\Delta x(t)) + q(t) x(t) = \lambda w(t) x(t)$$

to characterize both self-adjoint subspace extensions and self-adjoint operator extensions of the corresponding minimal subspaces. R. Koplatadze and S. Pinelas [5] have considered the difference equation

$$\Delta^2 u(k) + p(k) |u(\sigma(k))|^{\lambda} \operatorname{sign} u(\sigma(k)) = 0$$

and obtained an oscillation criteria of new type to have a positive solution.

The purpose of the second this paper is to establish sufficient conditions of the existence of at least one solution of the discrete boundary value problem $(1.7)-(1.9)$ which is an extension of Tisdell,[13] as the second-order difference boundary value problem . We construct an appropriate Green's function by means of which the existence result is based on Banach's fixed point theorem. We give an example to illustrate the main result.

2. The First Part: Application of Fixed Point Theory

First, we give a sufficient condition to obtain the existence and uniqueness of solutions to $(1.1)-(1.2)$ by using Banach fixed point theorem.

Theorem 2.1. *Let $S:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ be a nonlinear operator defined by* (1.4). *If*

$$\sum_{j=0}^{n-1} f^2(t_j,0) < \frac{1}{2h^2\|G\|_2^2} - L^2, \tag{2.1}$$

then $\|S\| = \sup_{\|x\|_2\le 1}\|Sx\|_2 < 1$ where

$$\|G\|_2^2 = \sum_{i=0}^{n}\sum_{j=0}^{n-1} G^2(i,j) = \frac{1}{(u+v)^2}\sum_{i=0}^{n}\left(u^2 i + v^2(n-i)\right). \tag{2.2}$$

Moreover $(1.1)-(1.2)$ *has a unique solution*

Proof. The definitions of nonlinear operator S and the function g imply that

$$(Sx)_i = h\sum_{j=0}^{n-1} G(i,j)\, g(t_j,x_j) + h\sum_{j=0}^{n-1} G(i,j)\, f(t_j,0)$$

Taking the square of this expression and using Cauchy inequality we rewrite the sum as the following form

$$(Sx)_i^2 \le 2h^2\left(\sum_{j=0}^{n-1} G(i,j)\, g(t_j,x_j)\right)^2 + 2h^2\left(\sum_{j=0}^{n-1} G(i,j)\, f(t_j,0)\right)^2.$$

Hölder inequality implies that

$$\begin{aligned}(Sx)_i^2 &\le 2h^2\sum_{j=0}^{n-1} G^2(i,j)\sum_{j=0}^{n-1} g^2(t_j,x_j) + 2h^2\sum_{j=0}^{n-1} G^2(i,j)\sum_{j=0}^{n-1} f^2(t_j,0)\\ &= 2h^2\sum_{j=0}^{n-1} G^2(i,j)\left[\sum_{j=0}^{n-1} g^2(t_j,x_j) + \sum_{j=0}^{n-1} f^2(t_j,0)\right].\end{aligned}$$

From (1.6) we obtain

$$\begin{aligned}(Sx)_i^2 &\leq 2h^2\sum_{j=0}^{n-1}G^2(i,j)\left[L^2\sum_{j=0}^{n-1}x_j^2+\sum_{j=0}^{n-1}f^2(t_j,0)\right]\\ &\leq 2h^2\sum_{j=0}^{n-1}G^2(i,j)\left[L^2\|x\|_2^2+\sum_{j=0}^{n-1}f^2(t_j,0)\right].\end{aligned}$$

Hence we get

$$\|Sx\|_2\leq\sqrt{2}h\left[L^2\|x\|_2^2+\sum_{j=0}^{n-1}f^2(t_j,0)\right]^{1/2}\|G\|_2$$

Suppose that $\|x\|_2\leq 1$

$$\|Sx\|_2\leq\sqrt{2}h\left[L^2+\sum_{j=0}^{n-1}f^2(t_j,0)\right]^{1/2}\|G\|_2.$$

It follows that

$$\|S\|=\sup_{\|x\|_2\leq 1}\|Sx\|_2\leq\sqrt{2}h\left[L^2+\sum_{j=0}^{n-1}f^2(t_j,0)\right]^{1/2}\|G\|_2.$$

Using assumption (2.1) concludes $\|S\|<1$. Finally, setting $Tx=Sx+\frac{w}{u+v}$ implies that T is a contraction and (1.3) has a unique fixed point which is the solution to (1.1) – (1.2). □

Now we can express the iteration procedure. Let $x^{(0)}=\left(x_0^{(0)},x_2^{(0)},...,x_n^{(0)}\right)$ be an initial iteration and define the values $x^{(k)}=\left(x_0^{(k)},x_2^{(k)},...,x_n^{(k)}\right)$ as the kth iteration by the following recursive formula

$$x^{(k)}=Sx^{(k-1)}+\frac{w}{u+v}$$

here $S:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ is a nonlinear operator given by (1.4). Next we will give convergence analysis for the proposed iteration schemes.

Theorem 2.2. *Let $S:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ be the nonlinear operator defined by* (1.4). *If $f(t,x)$ is a Lipschitz continuous function satisfying* (2.1) *then for any initial guess $x^{(0)}=\left(x_0^{(0)},x_2^{(0)},...,x_n^{(0)}\right)$ the iteration*

$$x^{(k)}=Sx^{(k-1)}+\frac{w}{u+v}\tag{2.3}$$

converges to the solution of (1.3).

Proof. Subtracting (1.3) from (2.3) for $k=1$ we have

$$\left(x-x^{(1)}\right)_i=\left(Sx-Sx^{(0)}\right)_i=h\sum_{j=0}^{n-1}G(i,j)\left[f(t_j,x_j)-f\left(t_j,x_j^{(0)}\right)\right]$$

From Lipschitz property of the function f we have

$$\left(x - x^{(1)}\right)_i \leq hL \sum_{j=0}^{n-1} G(i,j) \left[x_j - x_j^{(0)}\right]$$

Using Hölder inequality implies

$$\begin{aligned}\left(x - x^{(1)}\right)_i &\leq hL \left(\sum_{j=0}^{n-1} G^2(i,j)\right)^{1/2} \left(\sum_{j=0}^{n-1} \left[x_j - x_j^{(0)}\right]^2\right)^{1/2} \\ &\leq hL \left\|x - x^{(0)}\right\|_2 \left(\sum_{j=0}^{n-1} G^2(i,j)\right)^{1/2}\end{aligned}$$

Taking the square and sum of this expression we obtain

$$\left\|x - x^{(1)}\right\|_2 \leq hL \|G\|_2 \left\|x - x^{(0)}\right\|_2$$

Similarly

$$\begin{aligned}\left\|x - x^{(2)}\right\|_2 &\leq hL \|G\|_2 \left\|x - x^{(1)}\right\|_2 \\ &= h^2 L^2 \|G\|_2^2 \left\|x - x^{(0)}\right\|_2\end{aligned}$$

and we now expand that

$$\left\|x - x^{(k)}\right\|_2 \leq h^k L^k \|G\|_2^k \left\|x - x^{(0)}\right\|_2, k = 0, 1, 2, ... \tag{2.4}$$

Since the function $f(t,x)$ satisfies the condition (2.1) we conclude that

$$\frac{1}{2h^2 \|G\|_2^2} - L^2 > 0$$

implies

$$Lh \|G\|_2 < 1$$

Hence from (2.4)

$$\lim_{k\to\infty} \left\|x - x^{(k)}\right\|_2 = 0$$

□

So far, we have proven the convergence of the recursive algorithm (2.3) to the fixed point of (1.3) .However, in practise, it is necessary to decide to stop the algorithm after a finite number of iterations. The following result shows us that iterations will be terminated as $\lim_{k\to\infty} \left\|x^{(k)} - x^{(k-1)}\right\|_2 = 0$

Theorem 2.3. *Let $S : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the nonlinear operator defined by* (1.4)*. If $f(t,x)$ is a Lipschitz continuous function satisfying* (2.1) *then*

$$\left\|x - x^{(k)}\right\|_2 \leq \frac{hL \|G\|_2}{1 - hL \|G\|_2} \left\|x^{(k)} - x^{(k-1)}\right\|_2$$

Proof. Subtracting (1.3) from (2.3) we have

$$\left(x - x^{(k)}\right)_i = \left(Sx - Sx^{(k-1)}\right)_i = h \sum_{j=0}^{n-1} G(i,j) \left[f(t_j, x_j) - f\left(t_j, x_j^{(k-1)}\right)\right]$$

From Lipschitz property of the function f we have

$$\left(x - x^{(k)}\right)_i \leq hL \sum_{j=0}^{n-1} G(i,j) \left[x_j - x_j^{(k-1)}\right]$$

Using Hölder inequality implies

$$\begin{aligned}\left(x - x^{(k)}\right)_i &\leq hL \left(\sum_{j=0}^{n-1} G^2(i,j)\right)^{1/2} \left(\sum_{j=0}^{n-1} \left[x_j - x_j^{(k-1)}\right]^2\right)^{1/2} \\ &\leq hL \left\|x - x^{(k-1)}\right\|_2 \left(\sum_{j=0}^{n-1} G^2(i,j)\right)^{1/2}\end{aligned}$$

Taking the square and sum of this expression we obtain

$$\begin{aligned}\left\|x - x^{(k)}\right\|_2 &\leq hL \|G\|_2 \left\|x - x^{(k-1)}\right\|_2 \\ &\leq hL \|G\|_2 \left(\left\|x - x^{(k)}\right\|_2 + \left\|x^{(k)} - x^{(k-1)}\right\|_2\right)\end{aligned}$$

Consequently, we have

$$\left\|x - x^{(k)}\right\|_2 \leq \frac{hL \|G\|_2}{1 - hL \|G\|_2} \left\|x^{(k)} - x^{(k-1)}\right\|_2$$

□

3. Numerical results for the first part

In order to test the method described above, we shall consider the following problems.

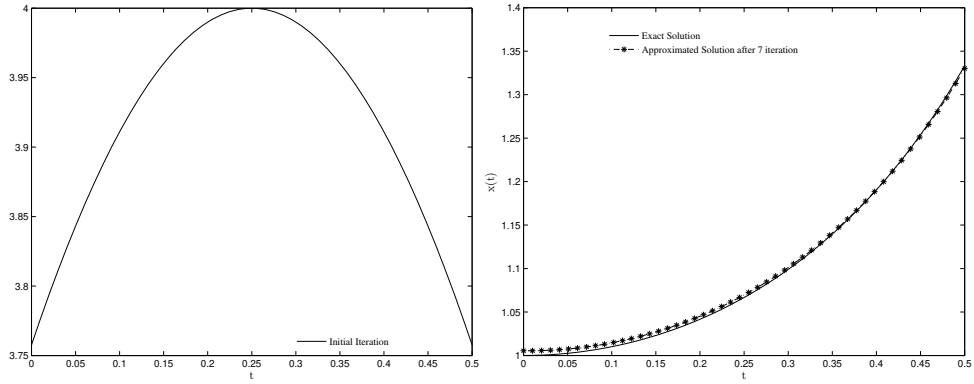
Example 3.1. *As the first example we consider the following boundary value problem*

$$x' = 2tx^2, 0 < t < 0.5$$
$$60x(0) + 100x(0.5) = \frac{580}{3}$$

with the exact solution $x(t) = 1/\left(1 - t^2\right)$ *and its discrete approximation*

$$\frac{\Delta x_i}{\tau} = 2t_i x_i^2, i = 0, 1, 2, ..., n$$
$$60x_0 + 100x_n = \frac{580}{3}$$

The step size $\tau = 0.5/n$ *and grid points* $t_i = i\tau, i = 0, 1, 2, ..., n$. *We try to identify the discrete solution* $x_i = 1/\left(1 - t_i^2\right)$ *from recursive formula* (2.3) *with an initial guess* $x^{(0)}(t) = 4\exp\left(-(t - 0.25)^2\right)$. *It possesses very small errors and provides suitable approximations, as seen in the figure.*



(A) (B)

FIGURE 1. (a) Initial guess $x^{(0)}(t) = 4\exp\left(-(t-0.25)^2\right)$; (b) The comparison for original and computed discrete solution $x_i = 1/\left(1-t_i^2\right)$ from recursive formula (2.3) with relative error $\frac{\max_i|x_i-x_i^h|}{\max_i|x_i|} * 100 = 0.4\%$

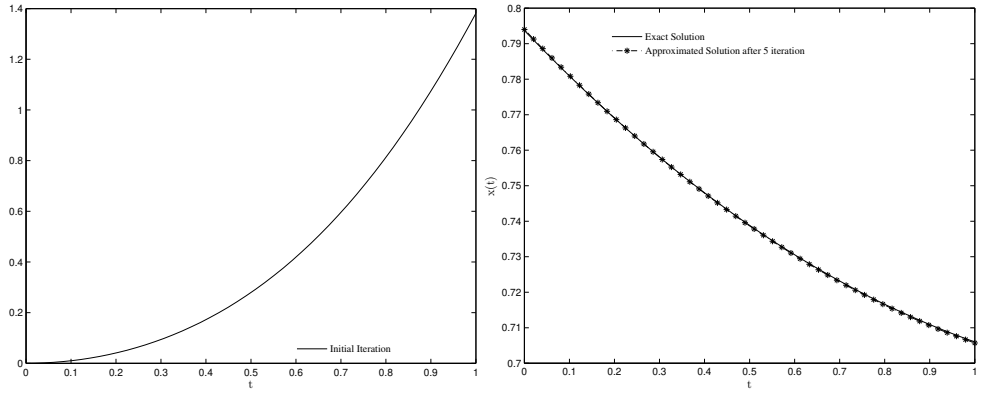
Example 3.2. *As the second example we consider the following boundary value problem*

$$x' = -\frac{\cos(t)\,x^4}{3}, 0 < t < 1$$
$$2x(0) + 2x(1) = 2.9994$$

with the exact solution $x(t) = 1/\sqrt[3]{2+\sin(t)}$ *and its discrete approximation*

$$\frac{\Delta x_i}{\tau} = -\frac{\cos(t_i)\,x_i^4}{3}, i = 0, 1, 2, ..., n$$
$$2x_0 + 2x_n = 2.9994$$

The step size $\tau = 1/n$ *and grid points* $t_i = i\tau, i = 0, 1, 2, ..., n$. *We try to identify the discrete solution* $x_i = 1/\sqrt[3]{2+\sin(t)}$ *from recursive formula* (2.3) *with an initial guess* $x^{(0)}(t) = t^2\cos t + t^3\sin t$. *Suitable approximation can be seen in the figure.*



(A) (B)

FIGURE 2. (a) Initial guess $x^{(0)}(t) = t^2 \cos t + t^3 \sin t$; (b) The comparison for original and computed discrete solution $x_i = 1/\sqrt[3]{2+\sin(t)}$ from recursive formula (2.3) with relative error $\frac{\max_i |x_i - x_i^h|}{\max_i |x_i|} * 100 = 0.03\%$

4. The Second Part: Existence and uniqueness results

A vector $x = (x_0, x_2, ..., x_n) \in \mathbb{R}^{n+1}$ which satisfies $(1.7) - (1.9)$ is called a solution of the discrete boundary value problem $(1.7) - (1.9)$. Let $\mathbb{R}^{n+1}$ be endowed with the norm

$$\|x\| = \sqrt{\sum_{i=0}^{n} x_i^2},\ x \in \mathbb{R}^{n+1}$$

First of all, we sum the equation (1.7) from 1 to i for any $i = 1, 2, \ldots, n$ we get

$$x_i - x_{i-1} = x_1 - x_0 + \tau^2 \sum_{j=1}^{i-1} f(t_j, x_j),\ 1 \le i \le n. \tag{4.1}$$

Replacing i by n in (4.1) and using the boundary condition (1.9) we obtain

$$\Delta x_0 = -\frac{\overline{v}\tau^2}{\overline{u}+\overline{v}} \sum_{j=1}^{n-1} f(t_j, x_j). \tag{4.2}$$

It follows from (4.2) and (4.1) that

$$x_i - x_{i-1} = -\frac{\overline{v}\tau^2}{\overline{u}+\overline{v}} \sum_{j=1}^{n-1} f(t_j, x_j) + \tau^2 \sum_{j=1}^{i-1} f(t_j, x_j),\ 1 \le i \le n, \tag{4.3}$$

and discrete Green's function can be constructed by given formula

$$G(i,j) = \begin{cases} \frac{\overline{u}}{\overline{u}+\overline{v}}, \text{ for } 1 \le j \le i-1 \\ -\frac{\overline{v}}{\overline{u}+\overline{v}}, \text{ for } i \le j \le n-1 \end{cases},\ 1 \le i \le n \tag{4.4}$$

consequently, by (4.3)

$$x_i - x_{i-1} = \tau^2 \sum_{j=1}^{n-1} G(i,j) f(t_j, x_j),\ 1 \le i \le n. \tag{4.5}$$

Next, summing (4.5) from 1 to i we obtain

$$x_i = x_0 + \tau^2 \sum_{k=1}^{i} \sum_{j=1}^{n-1} G(k,j) f(t_j, x_j), \ 0 \le i \le n. \tag{4.6}$$

putting $i = n$ and using the boundary condition (1.8) in the above formula gives

$$x_0 = -\frac{v\tau^2}{u+v} \sum_{k=1}^{n} \sum_{j=1}^{n-1} G(k,j) f(t_j, x_j).$$

This implies from (4.6) that

$$x_i = -\frac{v\tau^2}{u+v} \sum_{k=1}^{n} \sum_{j=1}^{n-1} G(k,j) f(t_j, x_j) + \tau^2 \sum_{k=1}^{i} \sum_{j=1}^{n-1} G(k,j) f(t_j, x_j).$$

So, we have

$$x_i = \tau^2 \sum_{j=1}^{n-1} \sum_{k=1}^{n} G^*(i,j,k) f(t_j, x_j), \ 0 \le i \le n$$

where $G^*(i,j,k)$ is Green's function defined by

$$G^*(i,j,k) = \begin{cases} \frac{u}{u+v} G(k,j), & \text{for } 1 \le k \le i \\ -\frac{v}{u+v} G(k,j), & \text{for } i+1 \le k \le n \end{cases}, \ 0 \le i \le n. \tag{4.7}$$

Now, consider the operator $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by

$$(Tx)_i = \tau^2 \sum_{j=1}^{n-1} \sum_{k=1}^{n} G^*(i,j,k) f(t_j, x_j), \ 0 \le i \le n. \tag{4.8}$$

Thus, our aim is to show that there exists at least one $x \in \mathbb{R}^{n+1}$ such that

$$x = Tx. \tag{4.9}$$

Theorem 4.1. *Let $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the nonlinear operator defined by* (4.8)*. If x is a fixed point of T then x is the solution of the discrete boundary value problem* (1.7) – (1.9)*.*

Proof. First assume that x is a fixed point of T. Then using the definition of $G^*(i,j,k)$ in (4.7) we get

$$x_i = -\frac{v\tau^2}{u+v} \sum_{k=1}^{n} \sum_{j=1}^{n-1} G(k,j) f(t_j, x_j) + \tau^2 \sum_{k=1}^{i} \sum_{j=1}^{n-1} G(k,j) f(t_j, x_j),$$

which satisfies (4.5). By using this identity we have

$$\begin{aligned} x_0 &= -\frac{v\tau^2}{u+v} \sum_{k=1}^{n} \sum_{j=1}^{n-1} G(k,j) f(t_j, x_j), \\ x_n &= \frac{u\tau^2}{u+v} \sum_{k=1}^{n} \sum_{j=1}^{n-1} G(k,j) f(t_j, x_j), \end{aligned}$$

and we obtain the boundary condition (1.8). From the definition of Green function given by (4.4) we obtain (4.3). Hence

$$\begin{aligned}
\Delta^2 x_i &= x_{i+1} - 2x_i + x_{i-1} \\
&= \tau^2 f(t_i, x_i), \\
x_1 - x_0 &= -\frac{\overline{v}\tau^2}{\overline{u}+\overline{v}} \sum_{j=1}^{n-1} f(t_j, x_j), \\
x_n - x_{n-1} &= \frac{\overline{u}\tau^2}{\overline{u}+\overline{v}} \sum_{j=1}^{n-1} f(t_j, x_j),
\end{aligned}$$

and (1.7), (1.9) are satisfied. □

Theorem 4.2. *Let $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the nonlinear operator defined by* (4.8). *If $f(t,x)$ is a Lipschitz continuous function and the following condition is satisfied*

$$L\tau^2 \sqrt{n \sum_{i=0}^{n-1} (a_i + b_i + c_i + d_i)^2} < 1$$

then the discrete boundary value problem $(1.7) - (1.9)$ *has a unique solution, where*

$$\begin{aligned}
a_i &= \left(\frac{u\overline{u}}{(u+v)(\overline{u}+\overline{v})}\right)^2 \frac{(i-1)\,i}{2}, \\
b_i &= \left(\frac{u\overline{v}}{(u+v)(\overline{u}+\overline{v})}\right)^2 \frac{i(2n-i-1)}{2}, \\
c_i &= \left(\frac{v\overline{u}}{(u+v)(\overline{u}+\overline{v})}\right)^2 \frac{n(n-1)-(i-1)\,i}{2}, \\
d_i &= \left(\frac{v\overline{v}}{(u+v)(\overline{u}+\overline{v})}\right)^2 \frac{(n-i-1)(n-i)}{2}.
\end{aligned}$$

Proof. Using (4.8) and Lipschitz property of $f(t,x)$, we get

$$|(Tx)_i - (Ty)_i| \le L\tau^2 \sum_{j=1}^{n-1} \sum_{k=1}^{n} |G^*(i,j,k)|\,|x_j - y_j|,$$

and applying Hölder inequality,

$$|(Tx)_i - (Ty)_i| \le L\tau^2 \left(\sum_{j=1}^{n-1} \left(\sum_{k=1}^{n} |G^*(i,j,k)|\right)^2\right)^{1/2} \left(\sum_{j=1}^{n-1} |x_j - y_j|^2\right)^{1/2}.$$

By Cauchy-Schwartz's inequality, we obtain

$$\begin{aligned}
|(Tx)_i - (Ty)_i| &\le L\tau^2 \left(\sum_{j=1}^{n-1} \sum_{k=1}^{n} n\,|G^*(i,j,k)|^2\right)^{1/2} \left(\sum_{j=1}^{n-1} |x_j - y_j|^2\right)^{1/2} \\
&\le L\tau^2 \sqrt{n}\,\|x-y\| \left(\sum_{j=1}^{n-1} \sum_{k=1}^{n} |G^*(i,j,k)|^2\right)
\end{aligned}$$

Hence, we conclude that

$$\|Tx - Ty\|^2 \leq L^2\tau^4 n \|x - y\|^2 \sum_{i=0}^{n-1} \left(\sum_{j=1}^{n-1} \sum_{k=1}^{n} |G^*(i,j,k)|^2 \right)^2 \tag{4.10}$$

Recalling the definition of $G^*(i,j,k)$ in (4.7) and the definition of Green function given by (4.4) we have

$$\begin{aligned}
\sum_{j=1}^{n-1}\sum_{k=1}^{n} |G^*(i,j,k)|^2 &= \sum_{j=1}^{n-1}\sum_{k=1}^{i} \left| \frac{u}{u+v} G(k,j) \right|^2 + \sum_{j=1}^{n-1}\sum_{k=i+1}^{n} \left| -\frac{v}{u+v} G(k,j) \right|^2 \\
&= \left(\frac{u}{u+v}\right)^2 \sum_{k=1}^{i}\sum_{j=1}^{n-1} |G(k,j)|^2 + \left(\frac{v}{u+v}\right)^2 \sum_{k=i+1}^{n}\sum_{j=1}^{n-1} |G(k,j)|^2 \\
&= \left(\frac{u}{u+v}\right)^2 \sum_{k=1}^{i} \left[\sum_{j=1}^{k-1} |G(k,j)|^2 + \sum_{j=k}^{n-1} |G(k,j)|^2 \right] \\
&+ \left(\frac{v}{u+v}\right)^2 \sum_{k=i+1}^{n} \left[\sum_{j=1}^{k-1} |G(k,j)|^2 + \sum_{j=k}^{n-1} |G(k,j)|^2 \right] \\
&= \left(\frac{u}{u+v}\right)^2 \sum_{k=1}^{i} \frac{(k-1)\,\overline{u}^2 + (n-k)\,\overline{v}^2}{(\overline{u}+\overline{v})^2} \\
&+ \left(\frac{v}{u+v}\right)^2 \sum_{k=i+1}^{n} \frac{(k-1)\,\overline{u}^2 + (n-k)\,\overline{v}^2}{(\overline{u}+\overline{v})^2} \\
&= \left(\frac{u\overline{u}}{(u+v)(\overline{u}+\overline{v})}\right)^2 \sum_{k=1}^{i} (k-1) + \left(\frac{u\overline{v}}{(u+v)(\overline{u}+\overline{v})}\right)^2 \sum_{k=1}^{i} (n-k) \\
&+ \left(\frac{v\overline{u}}{(u+v)(\overline{u}+\overline{v})}\right)^2 \sum_{k=i+1}^{n} (k-1) + \left(\frac{v\overline{v}}{(u+v)(\overline{u}+\overline{v})}\right)^2 \sum_{k=i+1}^{n} (n-k) \\
&= a_i + b_i + c_i + d_i
\end{aligned}$$

where

$$\begin{aligned}
a_i &= \left(\frac{u\overline{u}}{(u+v)(\overline{u}+\overline{v})}\right)^2 \frac{(i-1)\,i}{2}, \\
b_i &= \left(\frac{u\overline{v}}{(u+v)(\overline{u}+\overline{v})}\right)^2 \frac{i\,(2n-i-1)}{2}, \\
c_i &= \left(\frac{v\overline{u}}{(u+v)(\overline{u}+\overline{v})}\right)^2 \frac{n\,(n-1)-(i-1)\,i}{2}, \\
d_i &= \left(\frac{v\overline{v}}{(u+v)(\overline{u}+\overline{v})}\right)^2 \frac{(n-i-1)\,(n-i)}{2}.
\end{aligned}$$

Since $L\tau^2\sqrt{n\sum_{i=0}^{n-1}(a_i+b_i+c_i+d_i)^2} < 1$, (4.10) implies that the nonlinear operator defined by (4.8) is a contraction and possesses a unique fixed point by Banach Fixed Point theorem. Using Theorem 4.1, we conclude that the discrete boundary value problem (1.7) – (1.9) has a unique solution. □

5. Numerical results for the second part

In order to test the method described above, we shall consider the following problems.

Example 5.1. *As an example we consider the following boundary value problem:*

$$\begin{gathered} x''(t) = t^2 + x, t \in [0, 0.1], \\ x(0) = 0, \\ x'(0) = 0, \end{gathered}$$

with the exact solution $x(t) = e^t + e^{-t} - t^2 - 2$ *and its discrete approximation*

$$\begin{gathered} \frac{\Delta^2 x_i}{\tau^2} = f(t_i, x_i), i = 0, 1, 2, ..., n-1, \\ x_0 = 0, \\ \frac{\Delta x_0}{\tau} = 0, \end{gathered}$$

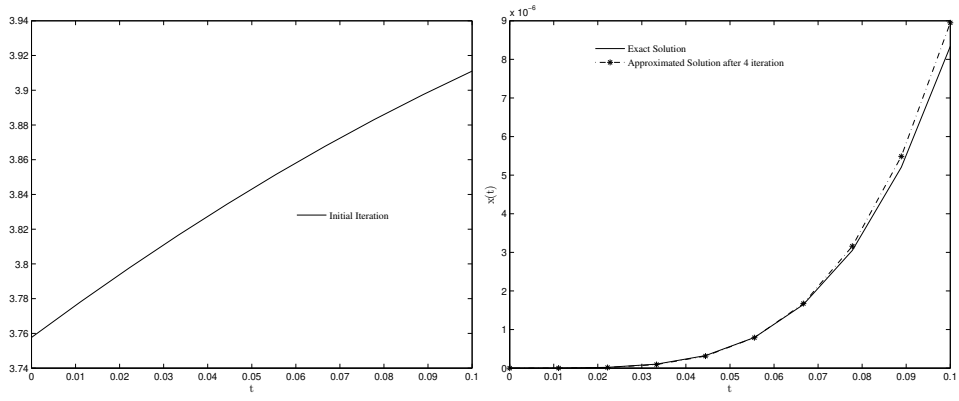
The step size $\tau = 0.1/n$ *and grid points* $t_i = i\tau, i = 0, 1, 2, ..., n$, *where* $n = 9$. *We try to identify the discrete solution* $x_i = e^{t_i} + e^{-t_i} - t_i^2 - 2$ *from* (4.9). *Here, the iteration procedure is expressed . Let* $x^{(0)} = \left(x_0^{(0)}, x_1^{(0)}, ..., x_n^{(0)}\right)$ *be an initial iteration and define the values* $x^{(m)} = \left(x_0^{(m)}, x_1^{(m)}, ..., x_n^{(m)}\right)$ *for the* m*th iteration by the following recursive formula*

$$x^{(m)} = Tx^{(m-1)},$$

here T *is a nonlinear operator given by* (4.8). *We set an initial guess as* $x^{(0)}(t) = 4\exp\left(-(t-0.25)^2\right)$. $f : [0, 0.1] \times \mathbb{R} \to \mathbb{R}$ *satisfies a Lipschitz condition with the Lipschitz constant* $L = 1$. *We obtain* $a_i,\ b_i,\ c_i,\ d_i,\ i = 0, ..., n-1$ *as follows:*

i	0	1	2	3	4	5	6	7	8
a_i	0	1	3	6	10	15	21	28	39
b_i	0	0	0	0	0	0	0	0	0
c_i	0	0	0	0	0	0	0	0	0
d_i	0	0	0	0	0	0	0	0	0

One can easily checked $L\tau^2\sqrt{n\sum_{i=0}^{n-1}(a_i + b_i + c_i + d_i)^2} = 0.021 < 1$. *So the condition for Theorem 4.2 is satisfied. Numerical experiment shows that the discrete solution possesses very small errors after* 4 *iteration and provides suitable approximations, as seen in the figure.*



(A) (B)

FIGURE 3. (a) Initial guess $x^{(0)}(t) = 4\exp\left(-(t-0.25)^2\right)$; (b) The comparison for original and computed discrete solution $x_i = e^{t_i} + e^{-t_i} - t_i^2 - 2$ from recursive formula (4.9) with the errror $\max_i \left|x_i - x_i^h\right| = 6.1 \times 10^{-7}$.

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Arzu Erdem Coşkun, Ali Demir
Kocaeli University, Faculty of Arts and Sciences, Department of Mathematics, Umuttepe Campus, 41380, Kocaeli - TURKEY
E-mail address: erdem.arzu@gmail.com, ademir@kocaeli.edu.tr

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