Turk. J. Math. Comput. Sci.
18(1)(2026) 192–208
© MatDer
DOI : 10.47000/tjmcs.1571950

On Error Bounds for Parameterized Trapezoid and Newton Formulas via Conformable Fractional Operators

Fatih Hezenci[1], Erhan Set[2,*], Hüseyin Budak[3]

[1]*Department of Mathematics, Faculty of Science and Arts, Duzce University, 81620 Duzce, Türkiye.*
[2]*Department of Mathematics, Faculty of Science and Arts, Ordu University, 52200 Ordu, Türkiye.*
[3]*Department of Mathematics, Faculty of Science and Arts, Kocaeli University, 41001 Kocaeli, Türkiye.*

Received: 22-10-2024 • Accepted: 21-11-2025

Abstract. In this paper, we are established that the parameterized inequalities reduce to some trapezoid and Newton-type inequalities with the help of the special choices. For this, first we present a parameterized integral identity inducing fractional integrals and then prove trapezoid and Newton-type inequalities for differentiable convex functions. These inequalities are proved by using the conformable fractional integrals. By using the special cases of our main results, we also give some new and previously obtained trapezoid and Newton-type inequalities.

2020 AMS Classification: 26D07, 26D10, 26D15

Keywords: Trapezoid inequality, Simpson's $\frac{3}{8}$ formula, conformable integrals, convex functions.

1. Introduction

Inequality theory and convex functions are closely related topics in mathematics. Convex functions play an important role in the study of inequalities, as many classic inequalities involve convex functions. For example, the Hermite–Hadamard-type inequalities and Newton-type inequalities both involve convex functions. Numerous mathematicians have studied these types of inequalities with great interest to generalise and extend them to the case of different classes of functions such as quasi-convex functions, log-convex, s-convex functions, etc.

Fractional calculus is a branch of mathematical analysis that generalizes the concepts of differentiation and integration to non-integer orders. It has applications in many fields, including physics, engineering, and finance. In fractional calculus, the order of differentiation or integration is a real or complex number rather than an integer. This allows for the study of systems that exhibit non-local behavior or long-range memory, which cannot be described by classical calculus. There are many different approaches to fractional calculus, including Riemann-Liouville, conformable, tempered, and Caputo. Fractional calculus has been used to model a wide range of physical systems, including viscoelastic materials, electrical circuits, and diffusion processes. It also has applications in signal processing, control theory, and data analysis.

İşcan and Wu [16] proved Hermite–Hadamard-type inequalities with the help of harmonic convexity. In paper [9], the authors established several Ostrowski's type inequalities to the case of differentiable mappings by using the k-fractional integrals. In paper [19], Khan et al. considered several new estimates of Hermite–Hadamard-type inequalities

*Corresponding Author
Email addresses: fatihezenci@gmail.com (F. Hezenci), erhanset@yahoo.com (E. Set), hsyn.budak@gmail.com (H. Budak)

for Riemann-Liouville fractional integrals and conformable fractional integrals. With the help of fractional integrals, Tunc [29] investigated several new variants of Hermite–Hadamard-type inequalities for h-convex functions.

Simpson's inequalities refer to a collection of mathematical inequalities that are derived from Simpson's rules. Many mathematicians have established Simpson's first rule, which is the basis of Simpson's quadrature formula. For example, in paper [26], some of Simpson's type inequalities are established by using the Riemann-Liouville fractional integrals. Furthermore, in paper [12], several fractional Simpson type inequalities are investigated for functions whose second derivatives in absolute value are convex. One can find additional important information on these type of inequalities, as well as various fractional integral operators in the articles [3, 4, 6, 7, 25, 32] and their reference lists.

The evaluation based on the three steps quadratic kernel in Simpson's second rule is known as the Newton-type results since it contains the rule of three-point Newton-Cotes quadrature. Inequalities that are derived using the Newton-type results are commonly referred to as Newton-type inequalities in the literature. Mathematicians have extensively studied this type of inequalities, For example, in paper [2] some new parameterized Newton-type inequalities are considered for differentiable convex functions in the setting of fractional calculus. In paper [10], several Newton-type inequalities are presented and some applications for special cases of real functions are also given. In addition, in paper [28], several Newton-type inequalities for differentiable convex functions are proved by using the Riemann-Liouville fractional integrals. The authors also give some inequalities of Riemann-Liouville fractional Newton-type for functions of bounded variation. Furthermore, some new integral inequalities of Newton-type for functions whose first derivative in absolute value at certain power are arithmetically-harmonically convex are obtained in paper [8]. For further important information and unexplained topics related to these type of inequalities, including convex differentiable functions, we suggest that the reader refers to Refs. [15, 21, 22, 24, 30, 33].

By utilizing previous research and the papers mentioned earlier, we aim to derive some parameterized inequalities for differentiable convex functions using conformable fractional integrals. In Sect. 2, the basic definitions of fractional calculus and other relevant research in this discipline are presented. In Sect. 3, we introduce a new parameterized integral equality involving conformable fractional integrals, based on which we provide some parameterized inequalities for differentiable convex functions. In addition, by using specific values of the parameters, we give some new and previously obtained trapezoid and Newton-type inequalities. In Sect. 4, we give some ideas for future work on this subject.

2. Definitions and Basic Properties

In this section, we will provide the basic definitions of Riemann-Liouville integrals and conformable integrals, which are widely used and well-known in the literature, to establish our main results. From the fact of fractional calculus theory, mathematical preliminaries will be given as follows:

The Hermite-Hadamard and Simpson-type inequalities have been investigated extensively in recent years. Simpson-type inequalities are inequalities that result from the application of the following rules, known as Simpson's rules:

i. Simpson's quadrature formula (Simpson's 1/3 rule):

$$\int_{\sigma}^{\delta} \mathcal{F}(x)\,dx \approx \frac{\delta-\sigma}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right].$$

ii. Simpson's second formula (Simpson's 3/8 rule)

$$\int_{\sigma}^{\delta} \mathcal{F}(x)\,dx \approx \frac{\delta-\sigma}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right].$$

Definition 2.1. For $0 < x, y < \infty$ and $x, y \in \mathbb{R}$, the gamma function, beta function, and incomplete beta function are described

$$\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t} dt,$$

$$\mathfrak{B}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt,$$

and

$$\mathscr{B}(x,y,r) := \int_0^r t^{x-1}(1-t)^{y-1}\,dt,$$

respectively.

In paper [20], Kilbas et al. presented fractional integrals, also called Riemann-Liouville integrals as follows:

Definition 2.2 ([11, 20]). For$\mathcal{F} \in L_1[\sigma,\delta]$, the Riemann-Liouville integrals $J^{\beta}_{\sigma+}\mathcal{F}(x)$ and $J^{\beta}_{\delta-}\mathcal{F}(x)$ of order $\beta > 0$ are given by

$$J^{\beta}_{\sigma+}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{\sigma}^{x}(x-t)^{\beta-1}\mathcal{F}(t)dt, \quad x > \sigma \tag{2.1}$$

and

$$J^{\beta}_{\delta-}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{x}^{\delta}(t-x)^{\beta-1}\mathcal{F}(t)dt, \quad x < \delta, \tag{2.2}$$

respectively. The Riemann-Liouville integrals coincide with the classical integrals for the case of $\beta = 1$.

In paper [18], Jarad et al. presented the conformable fractional integrals. They also provided certain characteristics and relationships between these operators and several other fractional operators in the literature. The conformable fractional integrals are defined as follows:

Definition 2.3 ([18]). For $\mathcal{F} \in L_1[\sigma,\delta]$, the fractional conformable integral operator ${}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(x)$ *and* ${}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(x)$ of order β in $C, Re(\beta) > 0$ and α in $(0,1]$ are given by

$${}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{\sigma}^{x}\left(\frac{(x-\sigma)^{\alpha}-(t-\sigma)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(t)}{(t-\sigma)^{1-\alpha}}dt, \quad t > \sigma \tag{2.3}$$

and

$${}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{x}^{\delta}\left(\frac{(\delta-x)^{\alpha}-(\delta-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(t)}{(\delta-t)^{1-\alpha}}dt, \quad t < \delta, \tag{2.4}$$

respectively.

Let us consider $\alpha = 1$ in equalities (2.3) and (2.4). Then, the fractional integral in (2.3) and (2.4) coincides with the Riemann-Liouville fractional integral in (2.1) and (2.2), respectively. Many research papers have been published on these topics, as documented in [1, 17, 23, 31] and the references cited therein.

3. Main Results of Parameterized Inequalities

Lemma 3.1. *Let us consider that $\mathcal{F} : [\sigma,\delta] \to \mathbb{R}$ is an absolutely continuous function on (σ,δ) so that $\mathcal{F}' \in L_1([\sigma,\delta])$. Then, the following equality holds:*

$$(1+\lambda_1-\lambda_3)\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2} + \left(\frac{\lambda_3-\lambda_1}{2}\right)\left[\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)\right] \tag{3.1}$$
$$-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma) + {}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right] = \frac{\alpha^{\beta}(\delta-\sigma)}{2}\sum_{i=1}^{3}I_i.$$

Here,

$$\begin{cases} I_1 = \int_0^{\frac{1}{3}}\left[\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta} - \frac{\lambda_1}{\alpha^{\beta}}\right]\left[\mathcal{F}'(t\delta+(1-t)\sigma) - \mathcal{F}'(t\sigma+(1-t)\delta)\right]dt, \\ I_2 = \int_{\frac{1}{3}}^{\frac{2}{3}}\left[\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta} - \frac{\lambda_2}{\alpha^{\beta}}\right]\left[\mathcal{F}'(t\delta+(1-t)\sigma) - \mathcal{F}'(t\sigma+(1-t)\delta)\right]dt, \\ I_3 = \int_{\frac{2}{3}}^{1}\left[\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta} - \frac{\lambda_3}{\alpha^{\beta}}\right]\left[\mathcal{F}'(t\delta+(1-t)\sigma) - \mathcal{F}'(t\sigma+(1-t)\delta)\right]dt. \end{cases}$$

Proof. From integration by parts, we obtain

$$
\begin{aligned}
I_1 &= \int_0^{\frac{1}{3}} \left[\left(\frac{1-(1-t)^\alpha}{\alpha} \right)^\beta - \frac{\lambda_1}{\alpha^\beta} \right] \left[\mathcal{F}'\left(t\delta + (1-t)\sigma\right) - \mathcal{F}'\left(t\sigma + (1-t)\delta\right) \right] dt \\
&= \frac{1}{\delta-\sigma} \left[\left(\frac{1-(1-t)^\alpha}{\alpha} \right)^\beta - \frac{\lambda_1}{\alpha^\beta} \right] \left[\mathcal{F}\left(t\delta + (1-t)\sigma\right) + \mathcal{F}\left(t\sigma + (1-t)\delta\right) \right] \Bigg|_0^{\frac{1}{3}} \\
&\quad - \frac{\beta}{\delta-\sigma} \int_0^{\frac{1}{3}} \left(\frac{1-(1-t)^\alpha}{\alpha} \right)^{\beta-1} (1-t)^{\alpha-1} \left[\mathcal{F}\left(t\delta + (1-t)\sigma\right) + \mathcal{F}\left(t\sigma + (1-t)\delta\right) \right] dt \\
&= \frac{1}{\delta-\sigma} \left[\left(\frac{1-\left(\frac{2}{3}\right)^\alpha}{\alpha} \right)^\beta - \frac{\lambda_1}{\alpha^\beta} \right] \left[\mathcal{F}\left(\frac{\sigma+2\delta}{3} \right) + \mathcal{F}\left(\frac{2\sigma+\delta}{3} \right) \right] + \frac{\lambda_1}{\alpha^\beta(\delta-\sigma)} \left[\mathcal{F}(\sigma) + \mathcal{F}(\delta) \right] \\
&\quad - \frac{\beta}{\delta-\sigma} \int_0^{\frac{1}{3}} \left(\frac{1-(1-t)^\alpha}{\alpha} \right)^{\beta-1} (1-t)^{\alpha-1} \left[\mathcal{F}\left(t\delta + (1-t)\sigma\right) + \mathcal{F}\left(t\sigma + (1-t)\delta\right) \right] dt. \qquad (3.2)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
I_2 &= \frac{1}{\delta-\sigma} \left[\left(\frac{1-\left(\frac{1}{3}\right)^\alpha}{\alpha} \right)^\beta - \frac{\lambda_2}{\alpha^\beta} \right] \left[\mathcal{F}\left(\frac{\sigma+2\delta}{3} \right) + \mathcal{F}\left(\frac{2\sigma+\delta}{3} \right) \right] \\
&\quad - \frac{1}{\delta-\sigma} \left[\left(\frac{1-\left(\frac{2}{3}\right)^\alpha}{\alpha} \right)^\beta - \frac{\lambda_2}{\alpha^\beta} \right] \left[\mathcal{F}\left(\frac{\sigma+2\delta}{3} \right) + \mathcal{F}\left(\frac{2\sigma+\delta}{3} \right) \right] \\
&\quad - \frac{\beta}{\delta-\sigma} \int_{\frac{1}{3}}^{\frac{2}{3}} \left(\frac{1-(1-t)^\alpha}{\alpha} \right)^{\beta-1} (1-t)^{\alpha-1} \left[\mathcal{F}\left(t\delta + (1-t)\sigma\right) + \mathcal{F}\left(t\sigma + (1-t)\delta\right) \right] dt
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \frac{1-\lambda_3}{\alpha^\beta(\delta-\sigma)} \left[\mathcal{F}(\sigma) + \mathcal{F}(\delta) \right] - \frac{1}{\delta-\sigma} \left[\left(\frac{1-\left(\frac{1}{3}\right)^\alpha}{\alpha} \right)^\beta - \frac{\lambda_3}{\alpha^\beta} \right] \left[\mathcal{F}\left(\frac{\sigma+2\delta}{3} \right) + \mathcal{F}\left(\frac{2\sigma+\delta}{3} \right) \right] \\
&\quad - \frac{\beta}{\delta-\sigma} \int_{\frac{2}{3}}^{1} \left(\frac{1-(1-t)^\alpha}{\alpha} \right)^{\beta-1} (1-t)^{\alpha-1} \left[\mathcal{F}\left(t\delta + (1-t)\sigma\right) + \mathcal{F}\left(t\sigma + (1-t)\delta\right) \right] dt. \qquad (3.3)
\end{aligned}
$$

If we combine equalities from (3.2) to (3.3), then we get

$$
\begin{aligned}
\sum_{i=1}^{3} I_i &= \frac{1}{\alpha^\beta(\delta-\sigma)} \left[(1+\lambda_1-\lambda_3) \left[\mathcal{F}(\sigma) + \mathcal{F}(\delta) \right] + (\lambda_3-\lambda_1) \left[\mathcal{F}\left(\frac{2\sigma+\delta}{3} \right) + \mathcal{F}\left(\frac{\sigma+2\delta}{3} \right) \right] \right] \\
&\quad - \frac{\beta}{\delta-\sigma} \int_0^1 \left(\frac{1-(1-t)^\alpha}{\alpha} \right)^{\beta-1} (1-t)^{\alpha-1} \left[\mathcal{F}\left(t\delta + (1-t)\sigma\right) + \mathcal{F}\left(t\sigma + (1-t)\delta\right) \right] dt. \qquad (3.4)
\end{aligned}
$$

By using the equality (3.4) and using the change of variable $x = t\delta + (1-t)\sigma$ and $x = t\sigma + (1-t)\delta$ for $t \in [0,1]$ respectively, it can be rewritten as follows

$$
\sum_{i=1}^{3} I_i = \frac{1}{\alpha^\beta(\delta-\sigma)} \left[(1+\lambda_1-\lambda_3) \left[\mathcal{F}(\sigma) + \mathcal{F}(\delta) \right] + (\lambda_3-\lambda_1) \left[\mathcal{F}\left(\frac{2\sigma+\delta}{3} \right) + \mathcal{F}\left(\frac{\sigma+2\delta}{3} \right) \right] \right]
$$

$$
\begin{aligned}
&-\left(\frac{1}{\delta-\sigma}\right)^{\alpha\beta+1}\frac{\Gamma(\beta+1)}{\Gamma(\beta)}\int_{\sigma}^{\delta}\left(\frac{(\delta-\sigma)^{\alpha}-(\delta-x)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{(\delta-x)^{1-\alpha}}dx\\
&-\left(\frac{1}{\delta-\sigma}\right)^{\alpha\beta+1}\frac{\Gamma(\beta+1)}{\Gamma(\beta)}\int_{\sigma}^{\delta}\left(\frac{(\delta-\sigma)^{\alpha}-(x-\sigma)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{(x-\sigma)^{1-\alpha}}dx\\
&=\frac{1}{\alpha^{\beta}(\delta-\sigma)}\left[(1+\lambda_1-\lambda_3)\left[\mathcal{F}(\sigma)+\mathcal{F}(\delta)\right]+(\lambda_3-\lambda_1)\left[\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)\right]\right]\\
&-\frac{\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta+1}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right].
\end{aligned}
\tag{3.5}
$$

If we multiply the both sides of (3.5) by $\frac{\alpha^{\beta}(\delta-\sigma)}{2}$, then the equality (3.1) is obtained simultaneously. □

Remark 3.2. If we choose $\alpha=1$, then Lemma 3.1 reduces to [2, Lemma 1].

Remark 3.3. If we choose $\lambda_1=\lambda_2=\lambda_3=\frac{1}{2}$ in Lemma 3.1, then Lemma 3.1 is equal to

$$
\begin{aligned}
&\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\\
&=\frac{\alpha^{\beta}(\delta-\sigma)}{2}\int_0^1\left[\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]\left[\mathcal{F}'(t\delta+(1-t)\sigma)-\mathcal{F}'(t\sigma+(1-t)\delta)\right]dt\\
&=\frac{\alpha^{\beta}(\delta-\sigma)}{2}\int_0^1\left[\left(\frac{1-t^{\alpha}}{\alpha}\right)^{\beta}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right]\mathcal{F}'(t\sigma+(1-t)\delta)\,dt,
\end{aligned}
$$

which is established in [27, Lemma 2.1]. This new identities can help us to obtain the bounds of the trapezoidal formula for conformable fractional integrals.

Remark 3.4. Let us take $\lambda_1=\frac{1}{8}$, $\lambda_2=\frac{1}{2}$ and $\lambda_3=\frac{7}{8}$ in Lemma 3.1. Then, Lemma 3.1 becomes to

$$
\frac{1}{8}\left[\mathcal{F}(\sigma)+3\left[\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\mathcal{F}(\delta)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]=\frac{\alpha^{\beta}(\delta-\sigma)}{2}\sum_{i=1}^{3}I_i.
$$

This coincides with [14]. This identity helps us to get Newton-type inequalities for conformable fractional integrals.

Corollary 3.5. *For $\lambda_1=\lambda_2=0$ and $\lambda_3=1$, Lemma 3.1 reduces to*

$$
\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]=\frac{\alpha^{\beta}(\delta-\sigma)}{2}\sum_{i=1}^{3}I_i.
$$

Theorem 3.6. *Suppose that the assumptions of Lemma 3.1 hold. Suppose also that the function $|\mathcal{F}'|$ is convex on $[\sigma,\delta]$. Then, the following inequality*

$$
\begin{aligned}
&\left|(1+\lambda_1-\lambda_3)\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}+\left(\frac{\lambda_3-\lambda_1}{2}\right)\left[\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right|\\
&\leq\frac{\alpha^{\beta}(\delta-\sigma)}{2}\left(\Omega_1(\alpha,\beta,\lambda_1)+\Omega_2(\alpha,\beta,\lambda_2)+\Omega_3(\alpha,\beta,\lambda_3)\right)\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right]
\end{aligned}
$$

is valid. Here,

$$\begin{cases} \Omega_1(\alpha,\beta,\lambda_1) = \int_0^{\frac{1}{3}} \left| \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta - \frac{\lambda_1}{\alpha^\beta} \right| dt, \\ \Omega_2(\alpha,\beta,\lambda_2) = \int_{\frac{1}{3}}^{\frac{2}{3}} \left| \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta - \frac{\lambda_2}{\alpha^\beta} \right| dt, \\ \Omega_3(\alpha,\beta,\lambda_3) = \int_{\frac{2}{3}}^{1} \left| \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta - \frac{\lambda_3}{\alpha^\beta} \right| dt. \end{cases} \tag{3.6}$$

Proof. With the help of modulus, Lemma 3.1 becomes

$$\begin{aligned} &\left|(1+\lambda_1-\lambda_3)\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}+\left(\frac{\lambda_3-\lambda_1}{2}\right)\left[\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)\right]-\frac{\alpha^\beta\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right| \\ &\leq \frac{\alpha^\beta(\delta-\sigma)}{2}\left[\int_0^{\frac{1}{3}}\left|\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta-\frac{\lambda_1}{\alpha^\beta}\right|\left|\mathcal{F}'(t\delta+(1-t)\sigma)-\mathcal{F}'(t\sigma+(1-t)\delta)\right|dt\right. \\ &\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta-\frac{\lambda_2}{\alpha^\beta}\right|\left|\mathcal{F}'(t\delta+(1-t)\sigma)-\mathcal{F}'(t\sigma+(1-t)\delta)\right|dt \\ &\quad\left.+\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta-\frac{\lambda_3}{\alpha^\beta}\right|\left|\mathcal{F}'(t\delta+(1-t)\sigma)-\mathcal{F}'(t\sigma+(1-t)\delta)\right|dt\right]. \end{aligned} \tag{3.7}$$

From the fact that $|\mathcal{F}'|$ is convex, it follows

$$\begin{aligned} &\left|(1+\lambda_1-\lambda_3)\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}+\left(\frac{\lambda_3-\lambda_1}{2}\right)\left[\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)\right]\right. \\ &\quad\left.-\frac{\alpha^\beta\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right| \\ &\leq \frac{\alpha^\beta(\delta-\sigma)}{2}\left[\int_0^{\frac{1}{3}}\left|\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta-\frac{\lambda_1}{\alpha^\beta}\right|\left[t\left|\mathcal{F}'(\delta)\right|+(1-t)\left|\mathcal{F}'(\sigma)\right|+t\left|\mathcal{F}'(\sigma)\right|+(1-t)\left|\mathcal{F}'(\delta)\right|\right]dt\right. \\ &\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta-\frac{\lambda_2}{\alpha^\beta}\right|\left[t\left|\mathcal{F}'(\delta)\right|+(1-t)\left|\mathcal{F}'(\sigma)\right|+t\left|\mathcal{F}'(\sigma)\right|+(1-t)\left|\mathcal{F}'(\delta)\right|\right]dt \\ &\quad\left.+\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta-\frac{\lambda_3}{\alpha^\beta}\right|\left[t\left|\mathcal{F}'(\delta)\right|+(1-t)\left|\mathcal{F}'(\sigma)\right|+t\left|\mathcal{F}'(\sigma)\right|+(1-t)\left|\mathcal{F}'(\delta)\right|\right]dt\right] \\ &= \frac{\alpha^\beta(\delta-\sigma)}{2}\left(\Omega_1(\alpha,\beta,\lambda_1)+\Omega_2(\alpha,\beta,\lambda_2)+\Omega_3(\alpha,\beta,\lambda_3)\right)\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right]. \end{aligned}$$

This finishes the proof of Theorem 3.6. □

Remark 3.7. If we choose $\alpha = 1$ in Theorem 3.6, then Theorem 3.6 becomes to [2, Theorem 3].

Corollary 3.8. *Let us consider $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2}$ in Theorem 3.6. Then, the following trapezoid-type inequality holds:*

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right|$$
$$\leq\frac{\alpha^{\beta}(\delta-\sigma)}{2}\psi(\alpha,\beta)\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right].$$

Here,

$$\psi(\alpha,\beta)=\Omega_1\left(\alpha,\beta,\frac{1}{2}\right)+\Omega_2\left(\alpha,\beta,\frac{1}{2}\right)+\Omega_3\left(\alpha,\beta,\frac{1}{2}\right)$$
$$=\int_0^1\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|dt=\frac{1}{\alpha^{\beta}}\left[\frac{1}{2}-\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right]$$
$$+\frac{1}{\alpha^{\beta+1}}\left[\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)\right],$$

where $\mathfrak{B}$ and $\mathscr{B}$ denote the beta function and incomplete beta function, respectively.

Corollary 3.9. *Let us consider $\alpha = 1$, $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2}$ in Theorem 3.6. Then, the following trapezoid-type inequality holds:*

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{\Gamma(\beta+1)}{2(\delta-\sigma)^{\beta}}\left[J^{\beta}_{\sigma+}\mathcal{F}(\delta)+J^{\beta}_{\delta-}\mathcal{F}(\sigma)\right]\right|$$
$$\leq\frac{\delta-\sigma}{2}\left(\frac{\beta}{\beta+1}\left(\frac{1}{2}\right)^{\frac{1}{\beta}}+\frac{1-\beta}{2(\beta+1)}\right)\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right].$$

Remark 3.10. If we select $\beta = 1$ in Corollary 3.9, then Corollary 3.9 becomes

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(x)dx\right|\leq\frac{\delta-\sigma}{8}\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right].$$

We recapture the inequality established in [5, Theorem 2.2].

Remark 3.11. Let us note $\lambda_1 = \frac{1}{8}$, $\lambda_2 = \frac{1}{2}$, and $\lambda_3 = \frac{7}{8}$ in Theorem 3.6. Then, Theorem 3.6 reduces to the following Newton-type inequality

$$\left|\frac{1}{8}\left[\mathcal{F}(\sigma)+3\left[\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\mathcal{F}(\delta)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right|$$
$$\leq\frac{\alpha^{\beta}(\delta-\sigma)}{2}\left(\Omega_1\left(\alpha,\beta,\frac{1}{8}\right)+\Omega_2\left(\alpha,\beta,\frac{1}{2}\right)+\Omega_3\left(\alpha,\beta,\frac{7}{8}\right)\right)\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right],$$

which is established by Hezenci et al. in paper [14].

Remark 3.12. Let us note $\alpha = 1$, $\lambda_1 = \frac{1}{8}$, $\lambda_2 = \frac{1}{2}$, and $\lambda_3 = \frac{7}{8}$ in Theorem 3.6. Then, Theorem 3.6 reduces to the following Newton-type inequality

$$\left|\frac{1}{8}\left[\mathcal{F}(\sigma)+3\left[\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\mathcal{F}(\delta)\right]-\frac{\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha}}\left[J^{\beta}_{\sigma+}\mathcal{F}(\delta)+J^{\beta}_{\delta-}\mathcal{F}(\sigma)\right]\right|$$
$$\leq\frac{\delta-\sigma}{2}\left(\Omega_1\left(1,\beta,\frac{1}{8}\right)+\Omega_2\left(1,\beta,\frac{1}{2}\right)+\Omega_3\left(1,\beta,\frac{7}{8}\right)\right)\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right],$$

which is proved by [13, Theorem 3].

Remark 3.13. For $\beta = 1$ in Remark 3.12, the following inequality holds:

$$\left|\frac{1}{8}\left[\mathcal{F}(\sigma)+3\left[\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\mathcal{F}(\delta)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(t)\,dt\right|$$

$$\leq \frac{25(\delta-\sigma)^2}{576}\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right].$$

We recapture the inequality proved in [28, Remark 3].

Corollary 3.14. *If we consider $\lambda_1=\lambda_2=0$ and $\lambda_3=1$ in Theorem 3.6, then Theorem 3.6 coincides with*

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right|$$
$$\leq \frac{\delta-\sigma}{2\alpha}\left[\frac{\alpha}{3}-\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)+2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},1-\left(\frac{1}{3}\right)^{\alpha}\right)\right]\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right].$$

Corollary 3.15. *Let us consider $\alpha=1$, $\lambda_1=\lambda_2=0$, and $\lambda_3=1$ in Theorem 3.6. Then, Theorem 3.6 becomes*

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]-\frac{\Gamma(\beta+1)}{2(\delta-\sigma)^{\beta}}\left[J^{\beta}_{\sigma+}\mathcal{F}(\delta)+J^{\beta}_{\delta-}\mathcal{F}(\sigma)\right]\right|$$
$$\leq \frac{\delta-\sigma}{2}\left(\frac{\beta-2}{3(\beta+1)}+\frac{2}{\beta+1}\left(\frac{2}{3}\right)^{\beta+1}\right)\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right].$$

Corollary 3.16. *If we select $\beta=1$ in Corollary 3.15, then Corollary 3.15 reduces to*

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(t)\,dt\right|\leq\frac{5(\delta-\sigma)}{36}\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right].$$

Theorem 3.17. *Let us consider that the assumptions of Lemma 3.1 are valid and the function $|\mathcal{F}'|^q$, $q>1$ is convex on $[\sigma,\delta]$. Then, the following inequality holds:*

$$\left|(1+\lambda_1-\lambda_3)\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}+\left(\frac{\lambda_3-\lambda_1}{2}\right)\left[\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)\right]\right.$$
$$\left.-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right|$$
$$\leq\frac{\alpha^{\beta}(\delta-\sigma)}{2}\left[\left(\varphi_1^p(\alpha,\beta,\lambda_1)+\varphi_3^p(\alpha,\beta,\lambda_3)\right)\left[\left(\frac{5|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}'(\sigma)|^q+5|\mathcal{F}'(\delta)|^q}{18}\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+2\varphi_2^p(\alpha,\beta,\lambda_2)\left(\frac{|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{6}\right)^{\frac{1}{q}}\right].$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ and

$$\begin{cases}\varphi_1^p(\alpha,\beta,\lambda_1)=\left(\int_0^{\frac{1}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_1}{\alpha^{\beta}}\right|^p dt\right)^{\frac{1}{p}},\\ \varphi_2^p(\alpha,\beta,\lambda_2)=\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_2}{\alpha^{\beta}}\right|^p dt\right)^{\frac{1}{p}},\\ \varphi_3^p(\alpha,\beta,\lambda_3)=\left(\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_3}{\alpha^{\beta}}\right|^p dt\right)^{\frac{1}{p}}.\end{cases}$$

Proof. By applying Hölder inequality in inequality (3.7), it follows

$$\left|(1+\lambda_1-\lambda_3)\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}+\left(\frac{\lambda_3-\lambda_1}{2}\right)\left[\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)\right]\right.$$
$$\left.-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right|$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{2}\left[\left(\int_0^{\frac{1}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_1}{\alpha^{\beta}}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left|\mathcal{F}'\left(t\delta+(1-t)\sigma\right)\right|^{q}dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^{\frac{1}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_1}{\alpha^{\beta}}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left|\mathcal{F}'\left(t\sigma+(1-t)\delta\right)\right|^{q}dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_2}{\alpha^{\beta}}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathcal{F}'\left(t\delta+(1-t)\sigma\right)\right|^{q}dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_2}{\alpha^{\beta}}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathcal{F}'\left(t\sigma+(1-t)\delta\right)\right|^{q}dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_3}{\alpha^{\beta}}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\left|\mathcal{F}'\left(t\delta+(1-t)\sigma\right)\right|^{q}dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_3}{\alpha^{\beta}}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\left|\mathcal{F}'\left(t\sigma+(1-t)\delta\right)\right|^{q}dt\right)^{\frac{1}{q}}\right].$$

From the fact that $|\mathcal{F}'|^{q}$ is convex, we have

$$\left|(1+\lambda_1-\lambda_3)\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}+\left(\frac{\lambda_3-\lambda_1}{2}\right)\left[\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)\right]\right.$$
$$\left.-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right|$$
$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{2}\left[\left(\int_0^{\frac{1}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_1}{\alpha^{\beta}}\right|^{p}dt\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{3}}t\left|\mathcal{F}'(\delta)\right|^{q}+(1-t)\left|\mathcal{F}'(\sigma)\right|^{q}dt\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\int_0^{\frac{1}{3}}t\left|\mathcal{F}'(\sigma)\right|^{q}+(1-t)\left|\mathcal{F}'(\delta)\right|^{q}dt\right)^{\frac{1}{q}}\right]+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_2}{\alpha^{\beta}}\right|^{p}dt\right)^{\frac{1}{p}}$$
$$\times\left[\left(\int_{\frac{1}{3}}^{\frac{2}{3}}t\left|\mathcal{F}'(\delta)\right|^{q}+(1-t)\left|\mathcal{F}'(\sigma)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}t\left|\mathcal{F}'(\sigma)\right|^{q}+(1-t)\left|\mathcal{F}'(\delta)\right|^{q}dt\right)^{\frac{1}{q}}\right]$$
$$+\left(\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_3}{\alpha^{\beta}}\right|^{p}dt\right)^{\frac{1}{p}}\left[\left(\int_{\frac{2}{3}}^{1}t\left|\mathcal{F}'(\delta)\right|^{q}+(1-t)\left|\mathcal{F}'(\sigma)\right|^{q}dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{\frac{2}{3}}^{1} t\left|\mathcal{F}'(\sigma)\right|^q+(1-t)\left|\mathcal{F}'(\delta)\right|^q dt\right)^{\frac{1}{q}}\Bigg]\Bigg]$$

$$=\frac{\alpha^\beta(\delta-\sigma)}{2}\Bigg[\Bigg(\left(\int_0^{\frac{1}{3}}\left|\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta-\frac{\lambda_1}{\alpha^\beta}\right|^p dt\right)^{\frac{1}{p}}+\left(\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta-\frac{\lambda_3}{\alpha^\beta}\right|^p dt\right)^{\frac{1}{p}}\Bigg)$$

$$\times\left[\left(\frac{5\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+5\left|\mathcal{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}}\right]$$

$$+2\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta-\frac{\lambda_2}{\alpha^\beta}\right|^p dt\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{6}\right)^{\frac{1}{q}}\Bigg].$$

This ends the proof of Theorem 3.17. □

Remark 3.18. If we assign $\alpha=1$ in Theorem 3.17, then Theorem 3.17 coincides with [2, Theorem 5].

Corollary 3.19. *Let us consider $\lambda_1=\lambda_2=\lambda_3=\frac{1}{2}$ in Theorem 3.17. Then, the following trapezoid-type inequality holds:*

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{\alpha^\beta\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^\beta\mathcal{J}^\alpha_{\delta-}\mathcal{F}(\sigma)+{}^\beta\mathcal{J}^\alpha_{\sigma+}\mathcal{F}(\delta)\right]\right|$$

$$\leq\frac{\alpha^\beta(\delta-\sigma)}{2}\left[\left(\varphi_1^p\left(\alpha,\beta,\frac{1}{2}\right)+\varphi_3^p\left(\alpha,\beta,\frac{1}{2}\right)\right)\left[\left(\frac{5\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+5\left|\mathcal{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+2\varphi_2^p\left(\alpha,\beta,\frac{1}{2}\right)\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{6}\right)^{\frac{1}{q}}\right].$$

Corollary 3.20. *If it is chosen $\alpha=1$ in Corollary 3.19, then the following trapezoid-type inequality holds:*

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{\Gamma(\beta+1)}{2(\delta-\sigma)^\beta}\left[J^\beta_{\sigma+}\mathcal{F}(\delta)+J^\beta_{\delta-}\mathcal{F}(\sigma)\right]\right|$$

$$\leq\frac{\delta-\sigma}{2}\left[\left(\varphi_1^p\left(1,\beta,\frac{1}{2}\right)+\varphi_3^p\left(1,\beta,\frac{1}{2}\right)\right)\left[\left(\frac{5\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+5\left|\mathcal{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+2\varphi_2^p\left(1,\beta,\frac{1}{2}\right)\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{6}\right)^{\frac{1}{q}}\right].$$

Corollary 3.21. *Consider $\beta=1$ in Corollary 3.20. Then, we obtain*

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{1}{\delta-\sigma}\int_\sigma^\delta\mathcal{F}(t)\,dt\right|$$

$$\leq\frac{\delta-\sigma}{9}\left[\left(\frac{3^{p+1}-1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{5\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+5\left|\mathcal{F}'(\delta)\right|^q}{6}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+\left(\frac{1}{2^p(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{2}\right)^{\frac{1}{q}}\right].$$

Remark 3.22. If we select $\lambda_1=\frac{1}{8}$, $\lambda_2=\frac{1}{2}$, and $\lambda_3=\frac{7}{8}$ in Theorem 3.17, then the following Newton-type inequality

$$\left|\frac{1}{8}\left[\mathcal{F}(\sigma)+3\left[\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\mathcal{F}(\delta)\right]-\frac{\alpha^\beta\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^\beta\mathcal{J}^\alpha_{\delta-}\mathcal{F}(\sigma)+{}^\beta\mathcal{J}^\alpha_{\sigma+}\mathcal{F}(\delta)\right]\right|$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{2}\left[\left(\varphi_1^p\left(\alpha,\beta,\frac{1}{8}\right)+\varphi_3^p\left(\alpha,\beta,\frac{7}{8}\right)\right)\left[\left(\frac{5\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+5\left|\mathcal{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+2\varphi_2^p\left(\alpha,\beta,\frac{1}{2}\right)\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{6}\right)^{\frac{1}{q}}\right],$$

which is given in paper [14].

Remark 3.23. If we assign $\alpha=1$, $\lambda_1=\frac{1}{8}$, $\lambda_2=\frac{1}{2}$, and $\lambda_3=\frac{7}{8}$ in Theorem 3.17, then the following Newton-type inequality

$$\left|\frac{1}{8}\left[\mathcal{F}(\sigma)+3\left[\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\mathcal{F}(\delta)\right]-\frac{\Gamma(\beta+1)}{2(\delta-\sigma)^{\beta}}\left[J_{\sigma+}^{\beta}\mathcal{F}(\delta)+J_{\delta-}^{\beta}\mathcal{F}(\sigma)\right]\right|$$
$$\leq \frac{\delta-\sigma}{2}\left[\left(\varphi_1^p\left(1,\beta,\frac{1}{8}\right)+\varphi_3^p\left(1,\beta,\frac{7}{8}\right)\right)\left[\left(\frac{5\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+5\left|\mathcal{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+2\varphi_2^p\left(1,\beta,\frac{1}{2}\right)\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{6}\right)^{\frac{1}{q}}\right],$$

which is given by [13, Theorem 4].

Remark 3.24. Let us note $\beta=1$ in Remark 3.23. Then, the following Newton-type inequality holds:

$$\left|\frac{1}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(t)\,dt\right|$$
$$\leq \frac{\delta-\sigma}{9}\left[\left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{5\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+5\left|\mathcal{F}'(\delta)\right|^q}{6}\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+\left(\frac{1}{2^p(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{2}\right)^{\frac{1}{q}}\right],$$

which is established in [28, Remark 5].

Corollary 3.25. *If we take $\lambda_1=\lambda_2=0$ and $\lambda_3=1$ in Theorem 3.17, then we have*

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{\delta-}^{\alpha}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}_{\sigma+}^{\alpha}\mathcal{F}(\delta)\right]\right|$$
$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{2}\left[\left(\varphi_1^p(\alpha,\beta,0)+\varphi_3^p(\alpha,\beta,1)\right)\left[\left(\frac{5\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+5\left|\mathcal{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+2\varphi_2^p(\alpha,\beta,0)\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{6}\right)^{\frac{1}{q}}\right].$$

Corollary 3.26. *Let us note $\alpha=1$ in Corollary 3.25. Then, we obtain*

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]-\frac{\Gamma(\beta+1)}{2(\delta-\sigma)^{\beta}}\left[J_{\sigma+}^{\beta}\mathcal{F}(\delta)+J_{\delta-}^{\beta}\mathcal{F}(\sigma)\right]\right|$$
$$\leq \frac{\delta-\sigma}{2}\left[\left(\varphi_1^p(1,\beta,0)+\varphi_3^p(1,\beta,1)\right)\left[\left(\frac{5\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+5\left|\mathcal{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+2\varphi_2^p(1,\beta,0)\left(\frac{\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{6}\right)^{\frac{1}{q}}\right].$$

Corollary 3.27. *If we consider $\beta = 1$ in Corollary 3.26, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(t)\,dt\right| \\
&\leq \frac{\delta-\sigma}{9}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{5\left|\mathcal{F}'(\sigma)\right|^{q}+\left|\mathcal{F}'(\delta)\right|^{q}}{6}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'(\sigma)\right|^{q}+5\left|\mathcal{F}'(\delta)\right|^{q}}{6}\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{F}'(\sigma)\right|^{q}+\left|\mathcal{F}'(\delta)\right|^{q}}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Theorem 3.28. *Let us consider that the assumptions of Lemma 3.1 are provided. Let us also consider that the function $|\mathcal{F}'|^{q}$, $q \geq 1$ is convex on $[\sigma,\delta]$. Then, the following inequality holds:*

$$\begin{aligned}
&\left|(1+\lambda_1-\lambda_3)\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}+\left(\frac{\lambda_3-\lambda_1}{2}\right)\left[\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)\right]\right. \\
&\quad\left.-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right| \\
&\leq \frac{\alpha^{\beta}(\delta-\sigma)}{2}\left\{\left(\Omega_1(\alpha,\beta,\lambda_1)\right)^{1-\frac{1}{q}}\right. \\
&\times\left[\left(\Omega_4(\alpha,\beta,\lambda_1)\left|\mathcal{F}'(\delta)\right|^{q}+\left(\Omega_1(\alpha,\beta,\lambda_1)-\Omega_4(\alpha,\beta,\lambda_1)\right)\left|\mathcal{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\Omega_4(\alpha,\beta,\lambda_1)\left|\mathcal{F}'(\sigma)\right|^{q}+\left(\Omega_1(\alpha,\beta,\lambda_1)-\Omega_4(\alpha,\beta,\lambda_1)\right)\left|\mathcal{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right] \\
&\quad+\left(\Omega_2(\alpha,\beta,\lambda_2)\right)^{1-\frac{1}{q}} \\
&\times\left[\left(\Omega_5(\alpha,\beta,\lambda_2)\left|\mathcal{F}'(\delta)\right|^{q}+\left(\Omega_2(\alpha,\beta,\lambda_2)-\Omega_5(\alpha,\beta,\lambda_2)\right)\left|\mathcal{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\Omega_5(\alpha,\beta,\lambda_2)\left|\mathcal{F}'(\sigma)\right|^{q}+\left(\Omega_2(\alpha,\beta,\lambda_2)-\Omega_5(\alpha,\beta,\lambda_2)\right)\left|\mathcal{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right] \\
&\quad+\left(\Omega_3(\alpha,\beta,\lambda_3)\right)^{1-\frac{1}{q}} \\
&\times\left[\left(\Omega_6(\alpha,\beta,\lambda_3)\left|\mathcal{F}'(\delta)\right|^{q}+\left(\Omega_3(\alpha,\beta,\lambda_3)-\Omega_6(\alpha,\beta,\lambda_3)\right)\left|\mathcal{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right. \\
&\quad\left.\left.+\left(\Omega_6(\alpha,\beta,\lambda_3)\left|\mathcal{F}'(\sigma)\right|^{q}+\left(\Omega_3(\alpha,\beta,\lambda_3)-\Omega_6(\alpha,\beta,\lambda_3)\right)\left|\mathcal{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}$$

Here, $\Omega_1(\alpha,\beta,\lambda_1)$, $\Omega_2(\alpha,\beta,\lambda_2)$ and $\Omega_3(\alpha,\beta,\lambda_3)$ are defined in (3.6) and

$$\begin{cases}
\Omega_4(\alpha,\beta,\lambda_1)=\displaystyle\int_0^{\frac{1}{3}} t\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_1}{\alpha^{\beta}}\right|dt, \\
\Omega_5(\alpha,\beta,\lambda_2)=\displaystyle\int_{\frac{1}{3}}^{\frac{2}{3}} t\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_2}{\alpha^{\beta}}\right|dt, \\
\Omega_6(\alpha,\beta,\lambda_3)=\displaystyle\int_{\frac{2}{3}}^{1} t\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_3}{\alpha^{\beta}}\right|dt.
\end{cases}$$

Proof. By using the facts of power-mean inequality in inequality (3.7), we have

$$\begin{aligned}
&\left|(1+\lambda_1-\lambda_3)\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}+\left(\frac{\lambda_3-\lambda_1}{2}\right)\left[\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)\right]\right. \\
&\quad\left.-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\alpha^{\beta}(\delta-\sigma)}{2}\left\{\left(\int_0^{\frac{1}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_1}{\alpha^{\beta}}\right|dt\right)^{1-\frac{1}{q}}\right.\\
&\times\left(\int_0^{\frac{1}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_1}{\alpha^{\beta}}\right|\left|\mathcal{F}'\left(t\delta+(1-t)\sigma\right)\right|^{q}dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^{\frac{1}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_1}{\alpha^{\beta}}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_1}{\alpha^{\beta}}\right|\left|\mathcal{F}'\left(t\sigma+(1-t)\delta\right)\right|^{q}dt\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_2}{\alpha^{\beta}}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_2}{\alpha^{\beta}}\right|\left|\mathcal{F}'\left(t\delta+(1-t)\sigma\right)\right|^{q}dt\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_2}{\alpha^{\beta}}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_2}{\alpha^{\beta}}\right|\left|\mathcal{F}'\left(t\sigma+(1-t)\delta\right)\right|^{q}dt\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_3}{\alpha^{\beta}}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_3}{\alpha^{\beta}}\right|\left|\mathcal{F}'\left(t\delta+(1-t)\sigma\right)\right|^{q}dt\right)^{\frac{1}{q}}\\
&\left.+\left(\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_3}{\alpha^{\beta}}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_3}{\alpha^{\beta}}\right|\left|\mathcal{F}'\left(t\sigma+(1-t)\delta\right)\right|^{q}dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Using the fact that $|\mathcal{F}'|^{q}$ is convex, we obtain

$$
\begin{aligned}
&\left|\left(1+\lambda_1-\lambda_3\right)\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}+\left(\frac{\lambda_3-\lambda_1}{2}\right)\left[\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)\right]\right.\\
&\quad\left.-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathcal{F}(\delta)\right]\right|\\
&\leq\frac{\alpha^{\beta}(\delta-\sigma)}{2}\left\{\left(\int_0^{\frac{1}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_1}{\alpha^{\beta}}\right|dt\right)^{1-\frac{1}{q}}\right.\\
&\quad\times\left[\left(\int_0^{\frac{1}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_1}{\alpha^{\beta}}\right|\left[t\left|\mathcal{F}'(\delta)\right|^{q}+(1-t)\left|\mathcal{F}'(\sigma)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.\\
&\quad\left.\left.+\left(\int_0^{\frac{1}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_1}{\alpha^{\beta}}\right|\left[t\left|\mathcal{F}'(\sigma)\right|^{q}+(1-t)\left|\mathcal{F}'(\delta)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_2}{\alpha^{\beta}}\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_2}{\alpha^{\beta}}\right|\left[t\left|\mathcal{F}'(\delta)\right|^{q}+(1-t)\left|\mathcal{F}'(\sigma)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_2}{\alpha^{\beta}}\right|\left[t\left|\mathcal{F}'(\sigma)\right|^{q}+(1-t)\left|\mathcal{F}'(\delta)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]\\
&+\left(\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_3}{\alpha^{\beta}}\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_3}{\alpha^{\beta}}\right|\left[t\left|\mathcal{F}'(\delta)\right|^{q}+(1-t)\left|\mathcal{F}'(\sigma)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{\lambda_3}{\alpha^{\beta}}\right|\left[t\left|\mathcal{F}'(\sigma)\right|^{q}+(1-t)\left|\mathcal{F}'(\delta)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}
$$

Consequently, we obtain the desired result of Theorem 3.28. □

Remark 3.29. Let us consider $\alpha = 1$ in Theorem 3.28. Then, Theorem 3.28 reduces to [2, Theorem 4].

Corollary 3.30. *Let us note $\alpha = 1$ and $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2}$ in Theorem 3.28. Then, the following trapezoid-type inequality holds:*

$$
\begin{aligned}
&\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{\Gamma(\beta+1)}{2(\delta-\sigma)^{\beta}}\left[J^{\beta}_{\sigma+}\mathcal{F}(\delta)+J^{\beta}_{\delta-}\mathcal{F}(\sigma)\right]\right|\\
&\leq\frac{\delta-\sigma}{2}\left[\left(\Omega_1\left(1,\beta,\frac{1}{2}\right)\right)^{1-\frac{1}{q}}\left[\left(\Omega_4\left(1,\beta,\frac{1}{2}\right)\left|\mathcal{F}'(\delta)\right|^{q}+\left(\Omega_1\left(1,\beta,\frac{1}{2}\right)-\Omega_4\left(1,\beta,\frac{1}{2}\right)\right)\left|\mathcal{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right.\right.\\
&\quad\left.+\left(\Omega_4\left(1,\beta,\frac{1}{2}\right)\left|\mathcal{F}'(\sigma)\right|^{q}+\left(\Omega_1\left(1,\beta,\frac{1}{2}\right)-\Omega_4\left(1,\beta,\frac{1}{2}\right)\right)\left|\mathcal{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right]\\
&\quad+\left(\Omega_2\left(1,\beta,\frac{1}{2}\right)\right)^{1-\frac{1}{q}}\left[\left(\Omega_5\left(1,\beta,\frac{1}{2}\right)\left|\mathcal{F}'(\delta)\right|^{q}+\left(\Omega_2\left(1,\beta,\frac{1}{2}\right)-\Omega_5\left(1,\beta,\frac{1}{2}\right)\right)\left|\mathcal{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\Omega_5\left(1,\beta,\frac{1}{2}\right)\left|\mathcal{F}'(\sigma)\right|^{q}+\left(\Omega_2\left(1,\beta,\frac{1}{2}\right)-\Omega_5\left(1,\beta,\frac{1}{2}\right)\right)\left|\mathcal{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right]\\
&\quad+\left(\Omega_3\left(1,\beta,\frac{1}{2}\right)\right)^{1-\frac{1}{q}}\left[\left(\Omega_6\left(1,\beta,\frac{1}{2}\right)\left|\mathcal{F}'(\delta)\right|^{q}+\left(\Omega_3\left(1,\beta,\frac{1}{2}\right)-\Omega_6\left(1,\beta,\frac{1}{2}\right)\right)\left|\mathcal{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&\quad\left.\left.+\left(\Omega_6\left(1,\beta,\frac{1}{2}\right)\left|\mathcal{F}'(\sigma)\right|^{q}+\left(\Omega_3\left(1,\beta,\frac{1}{2}\right)-\Omega_6\left(1,\beta,\frac{1}{2}\right)\right)\left|\mathcal{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right]\right].
\end{aligned}
$$

Corollary 3.31. *If we assign $\beta = 1$ in Corollary 3.30, then we have the following trapezoid-type inequality*

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(t)\,dt\right|$$
$$\leq\frac{(\delta-\sigma)}{2}\left[\frac{2}{9}\left[\left(\frac{5\left|\mathcal{F}'(\delta)\right|^{q}+31\left|\mathcal{F}'(\sigma)\right|^{q}}{36}\right)^{\frac{1}{q}}+\left(\frac{5\left|\mathcal{F}'(\sigma)\right|^{q}+31\left|\mathcal{F}'(\delta)\right|^{q}}{36}\right)^{\frac{1}{q}}\right]+\frac{1}{18}\left(\frac{\left|\mathcal{F}'(\sigma)\right|^{q}+\left|\mathcal{F}'(\delta)\right|^{q}}{2}\right)^{\frac{1}{q}}\right].$$

Remark 3.32. Consider $\alpha = 1$ and $\lambda_1 = \frac{1}{8}$, $\lambda_2 = \frac{1}{2}$, and $\lambda_3 = \frac{7}{8}$ in Theorem 3.28. Then, the following Newton-type inequality holds:

$$\left|\frac{1}{8}\left[\mathcal{F}(\sigma)+3\left[\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\mathcal{F}(\delta)\right]-\frac{\Gamma(\beta+1)}{2(\delta-\sigma)^{\beta}}\left[J^{\beta}_{\sigma+}\mathcal{F}(\delta)+J^{\beta}_{\delta-}\mathcal{F}(\sigma)\right]\right|$$
$$\leq\frac{(\delta-\sigma)}{2}\left[\left(\Omega_1\left(1,\beta,\frac{1}{8}\right)\right)^{1-\frac{1}{q}}\left[\left(\Omega_4\left(1,\beta,\frac{1}{8}\right)\left|\mathcal{F}'(\delta)\right|^{q}+\left(\Omega_1\left(1,\beta,\frac{1}{8}\right)-\Omega_4\left(1,\beta,\frac{1}{8}\right)\right)\left|\mathcal{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\Omega_4\left(1,\beta,\frac{1}{8}\right)\left|\mathcal{F}'(\sigma)\right|^{q}+\left(\Omega_1\left(1,\beta,\frac{1}{8}\right)-\Omega_4\left(1,\beta,\frac{1}{8}\right)\right)\left|\mathcal{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right]$$
$$+\left(\Omega_2\left(1,\beta,\frac{1}{2}\right)\right)^{1-\frac{1}{q}}\left[\left(\Omega_5\left(1,\beta,\frac{1}{2}\right)\left|\mathcal{F}'(\delta)\right|^{q}+\left(\Omega_2\left(1,\beta,\frac{1}{2}\right)-\Omega_5\left(1,\beta,\frac{1}{2}\right)\right)\left|\mathcal{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\Omega_5\left(1,\beta,\frac{1}{2}\right)\left|\mathcal{F}'(\sigma)\right|^{q}+\left(\Omega_2\left(1,\beta,\frac{1}{2}\right)-\Omega_5\left(1,\beta,\frac{1}{2}\right)\right)\left|\mathcal{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right]$$
$$+\left(\Omega_3\left(1,\beta,\frac{7}{8}\right)\right)^{1-\frac{1}{q}}\left[\left(\Omega_6\left(1,\beta,\frac{7}{8}\right)\left|\mathcal{F}'(\delta)\right|^{q}+\left(\Omega_3\left(1,\beta,\frac{7}{8}\right)-\Omega_6\left(1,\beta,\frac{7}{8}\right)\right)\left|\mathcal{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\Omega_6\left(1,\beta,\frac{7}{8}\right)\left|\mathcal{F}'(\sigma)\right|^{q}+\left(\Omega_3\left(1,\beta,\frac{7}{8}\right)-\Omega_6\left(1,\beta,\frac{7}{8}\right)\right)\left|\mathcal{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right]\right],$$

which is established in [13, Theorem 5].

Remark 3.33. If we select $\beta = 1$ in Remark 3.32, then we obtain the following Newton-type inequality

$$\left|\frac{1}{8}\left[\mathcal{F}(\sigma)+3\left[\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\mathcal{F}(\delta)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(t)\,dt\right|$$
$$\leq\frac{(\delta-\sigma)}{2}\left[\frac{17}{32\cdot 9}\left[\left(\frac{251\left|\mathcal{F}'(\delta)\right|^{q}+973\left|\mathcal{F}'(\sigma)\right|^{q}}{72}\right)^{\frac{1}{q}}+\left(\frac{251\left|\mathcal{F}'(\sigma)\right|^{q}+973\left|\mathcal{F}'(\delta)\right|^{q}}{72}\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+\frac{1}{18}\left(\frac{\left|\mathcal{F}'(\sigma)\right|^{q}+\left|\mathcal{F}'(\delta)\right|^{q}}{2}\right)^{\frac{1}{q}}\right],$$

which is presented by [28, Remark 4].

Corollary 3.34. *Let us consider $\alpha = 1$ and $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 1$ in Theorem 3.28. Then, the following inequality holds:*

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right]-\frac{\Gamma(\beta+1)}{2(\delta-\sigma)^{\beta}}\left[J^{\beta}_{\sigma+}\mathcal{F}(\delta)+J^{\beta}_{\delta-}\mathcal{F}(\sigma)\right]\right|$$
$$\leq\frac{\delta-\sigma}{2}\left\{\left(\Omega_1(1,\beta,0)\right)^{1-\frac{1}{q}}\left[\left(\Omega_4(1,\beta,0)\left|\mathcal{F}'(\delta)\right|^{q}+\left(\Omega_1(1,\beta,0)-\Omega_4(1,\beta,0)\right)\left|\mathcal{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right.\right.$$
$$\left.\left.+\left(\Omega_4(1,\beta,0)\left|\mathcal{F}'(\sigma)\right|^{q}+\left(\Omega_1(1,\beta,0)-\Omega_4(1,\beta,0)\right)\left|\mathcal{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right]\right.$$

$$
\begin{aligned}
&+ \left(\Omega_2(1,\beta,0)\right)^{1-\frac{1}{q}} \left[\left(\Omega_5(1,\beta,0)\left|\mathcal{F}'(\delta)\right|^q + \left(\Omega_2(1,\beta,0) - \Omega_5(1,\beta,0)\right)\left|\mathcal{F}'(\sigma)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\Omega_5(1,\beta,0)\left|\mathcal{F}'(\sigma)\right|^q + \left(\Omega_2(1,\beta,0) - \Omega_5(1,\beta,0)\right)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right] \\
&+ \left(\Omega_3(1,\beta,1)\right)^{1-\frac{1}{q}} \left[\left(\Omega_6(1,\beta,1)\left|\mathcal{F}'(\delta)\right|^q + \left(\Omega_3(1,\beta,1) - \Omega_6(1,\beta,1)\right)\left|\mathcal{F}'(\sigma)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\Omega_6(1,\beta,1)\left|\mathcal{F}'(\sigma)\right|^q + \left(\Omega_3(1,\beta,1) - \Omega_6(1,\beta,1)\right)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right]\Bigg\}.
\end{aligned}
$$

Corollary 3.35. *If we select $\beta = 1$ in Corollary 3.34, then we have*

$$
\begin{aligned}
&\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right) + \mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)\right] - \frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(t)\,dt\right| \\
&\leq \frac{\delta-\sigma}{2}\left\{\frac{1}{9}\left[\left(\frac{2\left|\mathcal{F}'(\delta)\right|^q + 7\left|\mathcal{F}'(\sigma)\right|^q}{9}\right)^{\frac{1}{q}} + \left(\frac{2\left|\mathcal{F}'(\sigma)\right|^q + 7\left|\mathcal{F}'(\delta)\right|^q}{9}\right)^{\frac{1}{q}}\right]\right. \\
&\left. + \frac{1}{6}\left[\left(\frac{14\left|\mathcal{F}'(\delta)\right|^q + 13\left|\mathcal{F}'(\sigma)\right|^q}{27}\right)^{\frac{1}{q}} + \left(\frac{14\left|\mathcal{F}'(\sigma)\right|^q + 13\left|\mathcal{F}'(\delta)\right|^q}{27}\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}
$$

4. Conclusion

Some trapezoid and Newton type inequalities are proved for conformable fractional integrals. Moreover, we also give some new and previously obtained parameterized inequalities with the help of special cases of our main results. Furthermore, we also present an example with graph to illustrate the special choice of basic result.

In the forthcoming papers, it is important to note that equivalent inequalities can also be obtained using Hadamard, tempered, and Katugampola fractional operators as well as fractional operators with an exponential kernel. Future workers will be able to obtain comparable inequalities for multiple convexity types and coordinated convexity on fractals, which is a fascinating and novel challenge.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

Conceptualization, F.H. and H.B.; investigation, H.B.; methodology, F.H., E.S.; validation, F.H.; visualization, F.H. and H.B.; writing-original draft, F.H. and E.S.; writing-review and editing, E.S. and H.B. All authors read and approved the final manuscript.

References

[1] Abdeljawad, T. *On conformable fractional calculus*, J. Comput. Appl. Math., **279**(2015), 57–66.

[2] Ali, M.A., Goodrich, C.S., Budak, H., *Some new parameterized Newton-type inequalities for differentiable functions via fractional integrals* J. Inequal. Appl., **2023**(1)(2023), 1-17.

[3] Budak H., Hezenci F., Kara H. *On parametrized inequalities of Ostrowski and Simpson type for convex functions via generalized fractional integral*, Math. Methods Appl. Sci., **44**(17)(2021), 12522–12536.

[4] Budak, H., Hezenci, F., Kara, H. *On generalized Ostrowski, Simpson and Trapezoidal type inequalities for co–ordinated convex functions via generalized fractional integrals*, Adv. Difference Equ., **2021**(2021), 1–32.

[5] Dragomir, S.S. Agarwal, R.P., *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. lett., **11**(5)(1998), 91–95.

[6] Du, T.S., Yuan X.M., *On the parameterized fractal integral inequalities and related applications*, Chaos Solitons Fractals, **170**(2023), 113375.

[7] Du, T., Ai, D., *Katugampola fractional integral inequalities in multiplicative calculus with multiple parameters*, Fractals, **33**(09)(2025), 1–26.

[8] Erden, S., Iftikhar, S., Kumam, P., Awan, M.U., *Some Newton's like inequalities with applications*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas, **114**(4)(2020), 1–13.

[9] Farid G., Usman M., *Ostrowski type k-fractional integral inequalities for MT-convex and h-convex functions*, Nonlinear Funct. Anal. Appl., **22**(2017), 627–639.

[10] Gao, S., Shi, W., *On new inequalities of Newton's type for functions whose second derivatives absolute values are convex*, Int. J. Pure Appl. Math., **74**(1)(2012), 33–41.

[11] Gorenflo R., Mainardi F., Fractional Calculus: Integral and Differential Equations of Fractional Order, Springer Verlag, Wien, 1997.

[12] Hezenci, F., Budak, H., Kara, H., *New version of fractional Simpson type inequalities for twice differentiable functions*, Adv. Difference Equ., **2021**, 2021.

[13] Hezenci, F., Budak, H., Kösem P., *A new version of Newton's inequalities for Riemann-Liouville fractional integrals*, Rocky Mountain J. Math., **53**(1)(2023), 49–64.

[14] Hezenci, F. Karagözoglu P., Budak, H., *Some error bounds for Newton formula in conformable fractional operators*, submitted.

[15] Iftikhar, S. Erden, S., Kumam, P., Awan, M.U., *Local fractional Newton's inequalities involving generalized harmonic convex functions*, *Adv. Difference Equ.*, **2020**(2020), 1–14.

[16] İşcan İ., Wu S., *Hermite–Hadamard type inequalities for harmonically convex functions via fractional integrals*, Appl. Math. Comput., **238**(2014), 237–244.

[17] Jarad, F., Abdeljawad, T., Baleanu, D., *On the generalized fractional derivatives and their Caputo modification*, J. Nonlinear Sci. Appl., **10**(5)(2017), 2607–2619.

[18] Jarad, F., Uğurlu E., Abdeljawad, T., Baleanu, D., *On a new class of fractional operators*, Adv. Difference Equ., **2017**(2017), 247.

[19] Khan, M.A., Iqbal, A., Suleman, M., Chu, Y.M., *Hermite–Hadamard type inequalities for fractional integrals via Green's function*, J. Inequal. Appl., 2018, **2018**, 1–15.

[20] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.

[21] Li, H., Lakhdari, A., Jarad, F. et al. *An expanded analysis of local fractional integral inequalities via generalized* (s,P)*-convexity*, J. Inequal. Appl., **2024**(2024) 78.

[22] Lakhdari, A., Benchettah, D.C., Meftah, B, *Fractional multiplicative Newton-type inequalities for multiplicative s-convex positive functions with application*, J. Comput. Appl. Math., **2025**465, 116600.

[23] Lakhdari, A., Mlaiki, N., Saleh, W., Abdeljawad, T., Meftah, B., *Exploring conformable fractional integral inequalities: A multi-parameter approach*, Fractals, **33**(07)(2025), 1–20.

[24] Noor, M.A., Noor, K.I., Iftikhar, S, *Some Newton's type inequalities for harmonic convex functions*, J. Adv. Math. Stud., **9**(1)(2016), 07–16.

[25] Park, J., *On some integral inequalities for twice differentiable Quasi–convex and convex functions via fractional integrals*, Appl. Math. Sci., **9**(62), **2015**, 3057–3069.

[26] Peng ,C., Zhou, C., Du, T.S., *Riemann-Liouville fractional Simpson's inequalities through generalized* (m,h_1,h_2)*-preinvexity*, Ital. J. Pure Appl. Math., **38**(2017), 345–367.

[27] Set, E., Choi, J., Gözpinar, A., *Hermite-Hadamard type inequalities involving nonlocal conformable fractional integrals*, Malaysian Journal of Mathematical Sciences, **15**(2021), 33-43.

[28] Sitthiwirattham, T., Nonlaopon, K., Ali, M.A., Budak, H., *Riemann-Liouville fractional Newton's type inequalities for differentiable convex functions*, Fractal Fract., **6**(3)(2022), 175.

[29] Tunc, M., *On new inequalities for h-convex functions via Riemann-Liouville fractional integration*, Filomat, **27**(2013), 559–565.

[30] Xu, H., Lakhdari, A., Jarad, F., Abdeljawad, T., Meftah, B., *On multiparametrized integral inequalities via generalized* α*-convexity on fractal set*, Math. Methods Appl. Sci., **48**(1)(2025), 980–1002.

[31] Xu, H., Awan, M.U., Meftah, B., Jarad, F., Lakhdari, A., *On conformable fractional Newton-type inequalities*., Fractals, **33**(07)(2025), 1–16.

[32] You, X., Hezenci, F., Budak, H., Kara, H., *New Simpson type inequalities for twice differentiable functions via generalized fractional integrals*, AIMS Mathematics, **7**(3)(2021), 3959–3971.

[33] Zhang, L.L., Peng, Y., Du, T.S., *On multiplicative Hermite-Hadamard- and Newton-type inequalities for multiplicatively (P;m)-convex functions*, J. Math. Anal. Appl. **534**(2)(2024), 128117.