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Hermite-Hadamard-Type Inequalities for Co-ordinated Convex Mappings via Generalized Fractional Integrals

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Abstract
In this investigation, Hermite-Hadamard-type inequalities are acquired for co-ordinated convex functions using generalized fractional integrals. These integrals include several well-known fractional operators, such as the Riemann-Liouville, Hadamard, and Katugampola fractional integrals. The findings of this study extend and generalize various inequalities from the existing literature. Moreover, by choosing specific parameter values, the established inequalities reduce to known results as special cases. In addition to these generalizations, new inequalities are established. To demonstrate the applicability of the results, examples are given, and graphical representations are utilized to support their validity.

Keywords: *Co-ordinated convex function,Fractional integral, Hermite-Hadamard type inequalities*

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1. Introduction and Preliminaries

A significant result in convex analysis is the Hermite-Hadamard inequality, which gives crucial boundaries for the integral mean of convex functions. It proves that the integral over a certain interval for any convex function is contained in the values of its midpoint and boundaries. In addition to establishing convexity, this inequality is the basis of several mathematical analysis generalizations and improvements. Midpoint-type inequalities improve accuracy by including midpoint analysis, while trapezoid-type inequalities provide better integral approximations utilizing boundary values. By increasing error estimates, other extensions, including Milne-type inequalities, Bullen-type inequalities, and refined Hermite-Hadamard bounds, improve numerical integration. The scientific basis of convex analysis is strengthened by these inequalities, which are also important in applied mathematics, especially in the areas of optimization, functional inequalities, and numerical approximation techniques.

The inequality attributed to C. Hermite and J. Hadamard (see, e.g., [1], [2, p.137]) is one of the most fundamental results in the theory of convex functions, possessing both a geometric interpretation and numerous applications. This inequality asserts that if $F : I \to \mathbb{R}$ is a convex function defined on an interval I of real numbers and if $\kappa, \sigma \in I$ with $\kappa < \sigma$, then the following holds:

$$F\left(\frac{\kappa+\sigma}{2}\right) \leq \frac{1}{\sigma-\kappa}\int_{\kappa}^{\sigma} F(\varpi)\varsigma\varpi \leq \frac{F(\kappa)+F(\sigma)}{2}. \tag{1.1}$$

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For concave functions, both inequalities hold in the reverse order. The Hermite-Hadamard inequality can be regarded as a refinement of convexity and follows directly from Jensen's inequality. In recent years, this result has received considerable attention, leading to numerous refinements and generalizations. In [3], the authors establish the Hadamard inequality without relying on any derivative properties, requiring only the convexity of the function over a closed interval. Furthermore, if the existence of the integral is assumed, the convexity condition is further relaxed to Jensen convexity. In [4], the authors present several Hermite-Hadamard-type inequalities for h-convex functions defined on convex subsets of real or complex linear spaces. A simple proof and a novel generalization of the Hermite-Hadamard inequality are given in [5].

There have been numerous studies in the literature devoted to integral inequalities, reflecting the continuing interest in inequality theory. For instance, Notuyas et al. investigated some new Pólya–Szegö type integral inequalities involving the Riemann–Liouville fractional integral operator and applied them to obtain Chebyshev-type fractional integral inequalities concerning the product of functions and integrals [6]. Butt et al. presented a generalized Jensen-type inequality for generalized harmonically convex functions on fractal sets and established a Jensen–Mercer type inequality via local fractional integrals [7]. Moreover, Vivas-Cortés et al. derived new (p,q)–Hermite–Hadamard type inequalities for convex functions by using the notions of $(p,q)^{\pi_2}$-derivative and $(p,q)^{\pi_2}$-integral [8]. In addition, Mohammed et al. obtained new Hermite–Hadamard-type fractional inequalities involving the incomplete gamma functions [9].

Additionally, in [10], a fractional-type integral identity is used to derive new Hermite-Hadamard-type inequalities involving Riemann-Liouville fractional integrals for (m,h_1,h_2)-convex functions. In [11], refinements and generalizations of the Hermite-Hadamard inequality are obtained for convex functions defined on a bounded closed interval of real numbers. Moreover, in [12], the authors present generalized Hermite-Hadamard-type integral inequalities for functions whose third derivatives are s-convex.

Definition 1.1. The concept of *co-ordinated convexity* extends the classical notion of convex functions to functions of two variables. A function $\digamma:(\varpi,u)\times(\psi,v)\to\mathbb{R}$ is called *co-ordinated convex* on Δ if, for all $(\varpi,u),(\psi,v)\in\Delta$ and for any $t,s\in[0,1]$, the following inequality holds:

$$\begin{aligned}\digamma(t\varpi+(1-t)\psi,su+(1-s)v)\leq{}& ts\digamma(\varpi,u)+t(1-s)\digamma(\varpi,v)\\&+s(1-t)\digamma(\psi,u)+(1-t)(1-s)\digamma(\psi,v).\end{aligned}\tag{1.2}$$

A function $\digamma$ is said to be *co-ordinated concave* on Δ if inequality (1.2) holds in the reversed direction for all $(\varpi,u),(\psi,v)\in\Delta$ and $t,s\in[0,1]$.

In [13], Dragomir established the following inequalities, which represent a class of Hermite-Hadamard-type inequalities for co-ordinated convex functions defined on rectangular regions in the plane $\mathbb{R}^2$.

Theorem 1.1. *Let $\digamma:\Delta\to\mathbb{R}$ be a co-ordinated convex function. Then, the following inequalities hold:*

$$\begin{aligned}\digamma\left(\frac{\kappa+\sigma}{2},\frac{\varphi+\varsigma}{2}\right)&\leq\frac{1}{2}\left[\frac{1}{\sigma-\kappa}\int_\kappa^\sigma\digamma\left(\varpi,\frac{\varphi+\varsigma}{2}\right)\varsigma\varpi+\frac{1}{\varsigma-\varphi}\int_\varphi^\varsigma\digamma\left(\frac{\kappa+\sigma}{2},\psi\right)\varsigma\psi\right]\\&\leq\frac{1}{(\sigma-\kappa)(\varsigma-\varphi)}\int_\kappa^\sigma\int_\varphi^\varsigma\digamma(\varpi,\psi)\,\varsigma\psi\varsigma\varpi\\&\leq\frac{1}{4}\left[\frac{1}{\sigma-\kappa}\int_\kappa^\sigma\digamma(\varpi,\varphi)\varsigma\varpi+\frac{1}{\sigma-\kappa}\int_\kappa^\sigma\digamma(\varpi,\varsigma)\varsigma\varpi\right.\\&\quad\left.+\frac{1}{\varsigma-\varphi}\int_\varphi^\varsigma\digamma(\kappa,\psi)\varsigma\psi+\frac{1}{\varsigma-\varphi}\int_\varphi^\varsigma\digamma(\sigma,\psi)\varsigma\psi\right]\\&\leq\frac{\digamma(\kappa,\varphi)+\digamma(\kappa,\varsigma)+\digamma(\sigma,\varphi)+\digamma(\sigma,\varsigma)}{4}.\end{aligned}\tag{1.3}$$

The inequalities above are sharp. Moreover, if $\digamma$ is a co-ordinated concave function, then all inequalities in (1.3) *hold in the reverse order.*

For further studies on Hermite-Hadamard-type inequalities related to co-ordinated convex functions, the following works provide valuable insights. In [14], Hadamard-type inequalities are extended to s-convex mappings and s-convex functions in two variables, along with various applications. A refined version of the Hermite-Hadamard inequality for functions convex on the coordinates is presented in [15]. In [16], a monotonic non-decreasing

mapping associated with Hadamard-type inequalities in two variables is introduced, and several inequalities for Lipschitzian mappings in two variables are established. Moreover, in [17], Hermite-Hadamard-type inequalities for extended s-convex functions on the coordinates within a rectangular plane are examined. For further studies on the Hermite–Hadamard inequality, the reader may refer to references [18], [19], and [20].

In the following, we provide the well-established definition of Riemann-Liouville fractional integrals, which play a fundamental role in fractional calculus.

Definition 1.2 (see, [21])**.** Let $\digamma \in L_1[\kappa, \sigma]$. The left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$, denoted by $J^{\alpha}_{\kappa+}\digamma$ and $J^{\alpha}_{\sigma-}\digamma$, are defined as follows

$$J^{\alpha}_{\kappa+}\digamma(\varpi) = \frac{1}{\Gamma(\alpha)} \int_{\kappa}^{\varpi} (\varpi - t)^{\alpha-1} \digamma(t)\, \varsigma t, \quad \varpi > \kappa,$$

$$J^{\alpha}_{\sigma-}\digamma(\varpi) = \frac{1}{\Gamma(\alpha)} \int_{\varpi}^{\sigma} (t - \varpi)^{\alpha-1} \digamma(t)\, \varsigma t, \quad \varpi < \sigma.$$

Here, $\Gamma(\alpha)$ represents the Gamma function. Moreover, when $\alpha = 0$, these operators reduce to the identity mapping:

$$J^{0}_{\kappa+}\digamma(\varpi) = J^{0}_{\sigma-}\digamma(\varpi) = \digamma(\varpi).$$

For a more comprehensive discussion on Riemann-Liouville fractional integrals, the reader may refer to [22], [23], [24].

Sarikaya et al. [25] were the first to establish the following integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.2. *Let $\digamma : [\kappa, \sigma] \to \mathbb{R}$ be a positive function with $0 \le \kappa < \sigma$ and $\digamma \in L_1[\kappa, \sigma]$. If $\digamma$ is convex on $[\kappa, \sigma]$, then the following fractional integral inequalities hold*

$$\digamma\left(\frac{\kappa + \sigma}{2}\right) \le \frac{\Gamma(\alpha + 1)}{2(\sigma - \kappa)^{\alpha}} \left[J^{\alpha}_{\kappa+}\digamma(\sigma) + J^{\alpha}_{\sigma-}\digamma(\kappa)\right] \le \frac{\digamma(\kappa) + \digamma(\sigma)}{2}, \tag{1.4}$$

for $\alpha > 0$.

Moreover, Sarıkaya [26] established a Hermite-Hadamard-type inequality for co-ordinated convex functions involving Riemann-Liouville fractional integrals. Additionally, several recent studies have investigated Hermite Hadamard inequalities for functions of one and two variables using Riemann-Liouville fractional integrals.For instance, in [27], Hadamard-type inequalities are derived for k-fractional Riemann-Liouville integrals, leading to corresponding results for the standard fractional Riemann-Liouville integrals. Furthermore, in [28], the Montgomery identities for Riemann-Liouville fractional integrals are extended and subsequently utilized to establish new integral inequalities.

The Hadamard fractional integrals are defined as follows:

Definition 1.3. Let $\digamma \in L_1([\kappa, \sigma])$. The left- and right-sided Hadamard fractional integrals of order $\alpha > 0$, denoted by $\mathbf{H}^{\alpha}_{\kappa+}\digamma$ and $\mathbf{H}^{\alpha}_{\sigma-}\digamma$, are given by

$$\mathbf{H}^{\alpha}_{\kappa+}\digamma(\varpi) = \frac{1}{\Gamma(\alpha)} \int_{\kappa}^{\varpi} \left(\ln \frac{\varpi}{t}\right)^{\alpha-1} \frac{\digamma(t)}{t} \varsigma t, \quad \varpi > \kappa,$$

$$\mathbf{H}^{\alpha}_{\sigma-}\digamma(\varpi) = \frac{1}{\Gamma(\alpha)} \int_{\varpi}^{\sigma} \left(\ln \frac{t}{\varpi}\right)^{\alpha-1} \frac{\digamma(t)}{t} \varsigma t, \quad \varpi < \sigma.$$

Recently, significant attention has been devoted to Hermite-Hadamard inequalities associated with Hadamard fractional integrals. For further advancements and related results, [29] presents new Hermite-Hadamard inequalities formulated in terms of series representations involving Hadamard fractional integrals. In [30], Hermite-Hadamard type inequalities for convex functions are studied using Hadamard fractional integrals. Additionally, [31] establishes three fundamental Riemann-Liouville fractional integral identities for a twice differentiable mapping.

In the literature, generalized fractional integrals with respect to an auxiliary function have been introduced to extend classical fractional operators. The formal definition is given below:

Definition 1.4. Let $g:[\kappa,\sigma]\to\mathbb{R}$ be an increasing and positive monotone function on $(\kappa,\sigma]$ with a continuous derivative $g'(\varpi)$ on (κ,σ). The left- and right-sided fractional integrals of a function F with respect to g, denoted by $I^{\alpha}_{\kappa^+;g}F(\varpi)$ and $I^{\alpha}_{\sigma^-;g}F(\varpi)$, are defined for $\alpha>0$ as follows

$$I^{\alpha}_{\kappa^+;g}F(\varpi)=\frac{1}{\Gamma(\alpha)}\int_{\kappa}^{\varpi}\frac{g'(t)F(t)}{[g(\varpi)-g(t)]^{1-\alpha}}\varsigma t,\quad \varpi>\kappa,$$

$$I^{\alpha}_{\sigma^-;g}F(\varpi)=\frac{1}{\Gamma(\alpha)}\int_{\varpi}^{\sigma}\frac{g'(t)F(t)}{[g(t)-g(\varpi)]^{1-\alpha}}\varsigma t,\quad \varpi<\sigma.$$

These integrals extend classical fractional integration by incorporating the function g, providing greater flexibility in various applications.

Jleli and Samet established the following Hermite-Hadamard-type inequalities:

Theorem 1.3 (see [32]). *Let $g:[\kappa,\sigma]\to\mathbb{R}$ be an increasing and positive monotone function on $(\kappa,\sigma]$ with a continuous derivative $g'(\varpi)$ on (κ,σ). Suppose that $\alpha>0$ and that F is a convex function on $[\kappa,\sigma]$. Then, the following inequality holds*

$$\varphi\left(\frac{\kappa+\sigma}{2}\right)\le\frac{\Gamma(\alpha+1)}{4\left[g(\sigma)-g(\kappa)\right]^{\alpha}}\left[I^{\alpha}_{\kappa+;g}\Phi(\sigma)+I^{\alpha}_{\sigma-;g}\Phi(\kappa)\right]\le\frac{\varphi(\kappa)+F(\sigma)}{2},\tag{1.5}$$

where $\Phi(\varpi)=\varphi(\varpi)+\widetilde{\varphi}(\varpi)$ and $\widetilde{\varphi}(\varpi)=\varphi(\kappa+\sigma-\varpi)$ for all $\varpi\in[\kappa,\sigma]$.

Theorem 1.4 (see [33]). *Let $\alpha>0$ and let $g:[\kappa,\sigma]\to\mathbb{R}$ be an increasing and positive monotone function on $(\kappa,\sigma]$ with a continuous derivative $g'(\varpi)$ on (κ,σ). If F is a convex function on $[\kappa,\sigma]$, then the following Hermite-Hadamard-type inequality for generalized integrals holds*

$$\varphi\left(\frac{\kappa+\sigma}{2}\right)\le\frac{\Gamma(\alpha+1)}{2\Upsilon^{\alpha}_{g}(1)}\left[I^{\alpha}_{\kappa+;g}\Phi(\sigma)+I^{\alpha}_{\sigma-;g}\Phi(\kappa)\right]\le\frac{\varphi(\kappa)+F(\sigma)}{2},\tag{1.6}$$

where $\Phi(\varpi)=\varphi(\varpi)+\widetilde{\varphi}(\varpi)$, with $\widetilde{\varphi}(\varpi)=\varphi(\kappa+\sigma-\varpi)$ for all $\varpi\in[\kappa,\sigma]$.

Moreover, the function $\Upsilon^{\alpha}_{g}(t)$ is defined as:

$$\Upsilon^{\alpha}_{g}(t)=\left[g(\sigma)-g\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma\right)\right]^{\alpha}\left[g\left(\frac{2-t}{2}\kappa+\frac{t}{2}\sigma\right)-g(\kappa)\right]^{\alpha}.$$

Hadamard fractional integrals for functions of two variables are defined as follows:

Definition 1.5. Let $F\in L_1([\kappa,\sigma]\times[\varphi,\varsigma])$. The Hadamard fractional integrals $\mathbf{J}^{\alpha,\beta}_{\kappa+,\varphi+}F$, $\mathbf{J}^{\alpha,\beta}_{\kappa+,\varsigma-}F$, $\mathbf{J}^{\alpha,\beta}_{\sigma-,\varphi+}F$, and $\mathbf{J}^{\alpha,\beta}_{\sigma-,\varsigma-}F$ of order $\alpha,\beta>0$ with $\kappa,\varphi\ge0$ are defined as follows

$$\begin{aligned}&\mathbf{J}^{\alpha,\beta}_{\kappa+,\varphi+}F(\varpi,\psi)\\&\quad=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa}^{\varpi}\int_{\varphi}^{\psi}\left(\ln\frac{\varpi}{t}\right)^{\alpha-1}\left(\ln\frac{\psi}{s}\right)^{\beta-1}\frac{F(t,s)}{ts}\varsigma s\varsigma t,\quad \varpi>\kappa,\quad \psi>\varphi,\\&\mathbf{J}^{\alpha,\beta}_{\kappa+,\varsigma-}F(\varpi,\psi)\\&\quad=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa}^{\varpi}\int_{\psi}^{\varsigma}\left(\ln\frac{\varpi}{t}\right)^{\alpha-1}\left(\ln\frac{s}{\psi}\right)^{\beta-1}\frac{F(t,s)}{ts}\varsigma s\varsigma t,\quad \varpi>\kappa,\quad \psi<\varsigma,\\&\mathbf{J}^{\alpha,\beta}_{\sigma-,\varphi+}F(\varpi,\psi)\\&\quad=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\varpi}^{\sigma}\int_{\varphi}^{\psi}\left(\ln\frac{t}{\varpi}\right)^{\alpha-1}\left(\ln\frac{\psi}{s}\right)^{\beta-1}\frac{F(t,s)}{ts}\varsigma s\varsigma t,\quad \varpi<\sigma,\quad \psi>\varphi,\\&\mathbf{J}^{\alpha,\beta}_{\sigma-,\varsigma-}F(\varpi,\psi)\\&\quad=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\varpi}^{\sigma}\int_{\psi}^{\varsigma}\left(\ln\frac{t}{\varpi}\right)^{\alpha-1}\left(\ln\frac{\psi}{s}\right)^{\beta-1}\frac{F(t,s)}{ts}\varsigma s\varsigma t,\quad \varpi<\sigma,\quad \psi<\varsigma.\end{aligned}$$

Now, we give following generalized fractional integral operators:

Definition 1.6. Let $g:[\kappa,\sigma]\to\mathbb{R}$ be an increasing and positive monotone function on $(\kappa,\sigma]$ with a continuous derivative $g'(\varpi)$ on (κ,σ). Similarly, let $w:[\varphi,\varsigma]\to\mathbb{R}$ be an increasing and positive monotone function on $(\varphi,\varsigma]$ with a continuous derivative $w'(\psi)$ on (φ,ς). Suppose that $F\in L_1([\kappa,\sigma]\times[\varphi,\varsigma])$. The generalized fractional integral operators for functions of two variables are defined as follows

$$\begin{aligned}
&\mathcal{J}^{\alpha,\beta}_{\kappa+,\varphi+;g,w}F(\varpi,\psi)\\
&\qquad=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_\kappa^\varpi\int_\varphi^\psi\frac{g'(t)}{[g(\varpi)-g(t)]^{1-\alpha}}\frac{w'(s)}{[w(\psi)-w(s)]^{1-\beta}}F(t,s)\,\varsigma s\varsigma t,\quad \varpi>\kappa,\quad \psi>\varphi,\\
&\mathcal{J}^{\alpha,\beta}_{\kappa+,\varsigma-;g,w}F(\varpi,\psi)\\
&\qquad=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_\kappa^\varpi\int_\psi^\varsigma\frac{g'(t)}{[g(\varpi)-g(t)]^{1-\alpha}}\frac{w'(s)}{[w(s)-w(\psi)]^{1-\beta}}F(t,s)\,\varsigma s\varsigma t,\quad \varpi>\kappa,\quad \psi<\varsigma,\\
&\mathcal{J}^{\alpha,\beta}_{\sigma-,\varphi+;g,w}F(\varpi,\psi)\\
&\qquad=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_\varpi^\sigma\int_\varphi^\psi\frac{g'(t)}{[g(t)-g(\varpi)]^{1-\alpha}}\frac{w'(s)}{[w(\psi)-w(s)]^{1-\beta}}F(t,s)\,\varsigma s\varsigma t,\quad \varpi<\sigma,\quad \psi>\varphi,\\
&\mathcal{J}^{\alpha,\beta}_{\sigma-,\varsigma-;g,w}F(\varpi,\psi)\\
&\qquad=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_\varpi^\sigma\int_\psi^\varsigma\frac{g'(t)}{[g(t)-g(\varpi)]^{1-\alpha}}\frac{w'(s)}{[w(s)-w(\psi)]^{1-\beta}}F(t,s)\,\varsigma s\varsigma t,\quad \varpi<\sigma,\quad \psi<\varsigma.
\end{aligned}$$

Similar to the above definitions, we introduce the following integrals:

$$\begin{aligned}
\mathcal{J}^{\alpha}_{\kappa+;g}F\left(\varpi,\frac{\varphi+\varsigma}{2}\right)&=\frac{1}{\Gamma(\alpha)}\int_\kappa^\varpi\frac{g'(t)}{[g(\varpi)-g(t)]^{1-\alpha}}F\left(t,\frac{\varphi+\varsigma}{2}\right)\varsigma t,\quad \varpi>\kappa,\\
\mathcal{J}^{\alpha}_{\sigma-;g}F\left(\varpi,\frac{\varphi+\varsigma}{2}\right)&=\frac{1}{\Gamma(\alpha)}\int_\varpi^\sigma\frac{g'(t)}{[g(t)-g(\varpi)]^{1-\alpha}}F\left(t,\frac{\varphi+\varsigma}{2}\right)\varsigma t,\quad \varpi<\sigma,\\
\mathcal{J}^{\beta}_{\varphi+;w}F\left(\frac{\kappa+\sigma}{2},\psi\right)&=\frac{1}{\Gamma(\beta)}\int_\varphi^\psi\frac{w'(s)}{[w(\psi)-w(s)]^{1-\beta}}F\left(\frac{\kappa+\sigma}{2},\psi\right)\varsigma s,\quad \psi>\varphi,\\
\mathcal{J}^{\beta}_{\varsigma-;w}F\left(\frac{\kappa+\sigma}{2},\psi\right)&=\frac{1}{\Gamma(\beta)}\int_\varphi^\psi\frac{w'(s)}{[w(s)-w(\psi)]^{1-\beta}}F\left(\frac{\kappa+\sigma}{2},\psi\right)\varsigma s,\quad \psi<\varsigma.
\end{aligned}$$

If we choose $g(t)=\frac{t^\rho}{\rho}$ and $w(s)=\frac{s^\sigma}{\sigma}$ in Definition 1.6, then we obtain the following Katugampola fractional integrals for functions of two variables, similar to the definitions given by Yaldiz in [34]:

Definition 1.7. Let $F\in L_1([\kappa,\sigma]\times[\varphi,\varsigma])$. The Katugampola fractional integrals for functions of two variables are defined as follows:

$$\begin{aligned}
&{}^{\rho,\sigma}\mathcal{I}^{\alpha,\beta}_{\kappa+,\varphi+}F(\varpi,\psi)\\
&\qquad=\frac{\rho^{1-\alpha}\sigma^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_\kappa^\varpi\int_\varphi^\psi\frac{t^{\rho-1}}{[\varpi^\rho-t^\rho]^{1-\alpha}}\frac{s^{\sigma-1}}{[\psi^\sigma-s^\sigma]^{1-\beta}}F(t,s)\,\varsigma s\varsigma t,\quad \varpi>\kappa,\quad \psi>\varphi,\\
&{}^{\rho,\sigma}\mathcal{I}^{\alpha,\beta}_{\kappa+,\varsigma-}F(\varpi,\psi)\\
&\qquad=\frac{\rho^{1-\alpha}\sigma^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_\kappa^\varpi\int_\psi^\varsigma\frac{t^{\rho-1}}{[\varpi^\rho-t^\rho]^{1-\alpha}}\frac{s^{\sigma-1}}{[s^\sigma-\psi^\sigma]^{1-\beta}}F(t,s)\,\varsigma s\varsigma t,\quad \varpi>\kappa,\quad \psi<\varsigma,\\
&{}^{\rho,\sigma}\mathcal{I}^{\alpha,\beta}_{\sigma-,\varphi+}F(\varpi,\psi)\\
&\qquad=\frac{\rho^{1-\alpha}\sigma^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_\varpi^\sigma\int_\varphi^\psi\frac{t^{\rho-1}}{[t^\rho-\varpi^\rho]^{1-\alpha}}\frac{s^{\sigma-1}}{[\psi^\sigma-s^\sigma]^{1-\beta}}F(t,s)\,\varsigma s\varsigma t,\quad \varpi<\sigma,\quad \psi>\varphi,
\end{aligned}$$

$$
\begin{aligned}
&{}^{\rho,\sigma}\mathcal{I}^{\alpha,\beta}_{\sigma-,\varsigma-}F(\varpi,\psi)\\
&\qquad=\frac{\rho^{1-\alpha}\sigma^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_{\varpi}^{\sigma}\int_{\psi}^{\varsigma}\frac{t^{\rho-1}}{\left[t^{\rho}-\varpi^{\rho}\right]^{1-\alpha}}\frac{s^{\sigma-1}}{\left[s^{\sigma}-\psi^{\sigma}\right]^{1-\beta}}F(t,s)\,\varsigma s\varsigma t,\quad \varpi<\sigma,\quad \psi<\varsigma.
\end{aligned}
$$

The primary objective of this study is to establish Hermite-Hadamard-type integral inequalities for co-ordinated convex functions involving generalized fractional integrals. The findings presented in this work extend and generalize results from previous studies. In [33], Budak obtained Hermite-Hadamard-type inequalities for convex functions utilizing generalized fractional integrals. In this study, we derive new Hermite-Hadamard inequalities specifically for co-ordinated convex functions. Additionally, illustrative examples will be provided to support the theoretical results. The validity of the obtained inequalities will be further verified through graphical representations.

Throughout this paper, we introduce several notations and mathematical expressions that will be fundamental in formulating our main results and proving the derived inequalities. These notations play a crucial role in our analysis and will be consistently referenced in subsequent sections.

Let $F:\Delta=[\kappa,\sigma]\times[\varphi,\varsigma]\to\mathbb{R}$. The following functions are frequently used in our study:

Function	Definition
$\widetilde{F_1}(\varpi,\psi)$	$F(\kappa+\sigma-\varpi,\psi)$
$\widetilde{F_2}(\varpi,\psi)$	$F(\varpi,\varphi+\varsigma-\psi)$
$\widetilde{F_3}(\varpi,\psi)$	$F(\kappa+\sigma-\varpi,\varphi+\varsigma-\psi)$
$G(\varpi,\psi)$	$F(\varpi,\psi)+\widetilde{F_2}(\varpi,\psi)$
$H(\varpi,\psi)$	$F(\varpi,\psi)+\widetilde{F_1}(\varpi,\psi)$
$K(\varpi,\psi)$	$\widetilde{F_1}(\varpi,\psi)+\widetilde{F_3}(\varpi,\psi)$
$L(\varpi,\psi)$	$\widetilde{F_2}(\varpi,\psi)+\widetilde{F_3}(\varpi,\psi)$
$F(\varpi,\psi)$	$\widetilde{F_1}(\varpi,\psi)+\widetilde{F_2}(\varpi,\psi)+\widetilde{F_3}(\varpi,\psi)+F(\varpi,\psi)$ $=\frac{G(\varpi,\psi)+H(\varpi,\psi)+K(\varpi,\psi)+L(\varpi,\psi)}{2}$

(1.7)

for all $(\varpi,\psi)\in[\kappa,\sigma]\times[\varphi,\varsigma]$.

Throughout this study let $g:[\kappa,\sigma]\to\mathbb{R}$ be an increasing and positive function on $(\kappa,\sigma]$ with a continuous derivative $g'(\varpi)$ on (κ,σ) and let $w:[\varphi,\varsigma]\to\mathbb{R}$ be an increasing and positive monotone function on $(\varphi,\varsigma]$ with a continuous derivative $w'(\psi)$ on (φ,ς).

The following functions are defined

$$
\Lambda^{\alpha}_{g}(t)=\left[g(\sigma)-g\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma\right)\right]^{\alpha}+\left[g\left(\frac{2-t}{2}\kappa+\frac{t}{2}\sigma\right)-g(\kappa)\right]^{\alpha},\quad 0\le t\le 1,
$$

$$
\Delta^{\beta}_{w}(s)=\left[w(\varsigma)-w\left(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma\right)\right]^{\beta}+\left[w\left(\frac{2-s}{2}\varphi+\frac{s}{2}\varsigma\right)-w(\varphi)\right]^{\beta},\quad 0\le s\le 1.
$$

for $\alpha>0$ and $\beta>0$ Particularly, for $t=1$ and $s=1$, these functions reduce to

$$
\Lambda^{\alpha}_{g}(1)=\left[g(\sigma)-g\left(\frac{\kappa+\sigma}{2}\right)\right]^{\alpha}+\left[g\left(\frac{\kappa+\sigma}{2}\right)-g(\kappa)\right]^{\alpha},
$$

$$
\Delta^{\beta}_{w}(1)=\left[w(\varsigma)-w\left(\frac{\varphi+\varsigma}{2}\right)\right]^{\beta}+\left[w\left(\frac{\varphi+\varsigma}{2}\right)-w(\varphi)\right]^{\beta}.
$$

For the logarithmic case, i.e., when $g(t)=\ln t$ and $w(s)=\ln s$, the expressions become

$$
\Lambda^{\alpha}_{\ln}(t)=\left[\ln\left(\frac{2\sigma}{t\kappa+(2-t)\sigma}\right)\right]^{\alpha}+\left[\ln\left(\frac{(2-t)\kappa+t\sigma}{2\kappa}\right)\right]^{\alpha},
$$

$$\Delta_{\ln}^{\beta}(s)=\left[\ln\left(\frac{2\varsigma}{s\varphi+(2-s)\varsigma}\right)\right]^{\beta}+\left[\ln\left(\frac{(2-s)\varphi+s\varsigma}{2\varphi}\right)\right]^{\beta}.$$

When $t=1$ and $s=1$, these simplify to

$$\Lambda_{\ln}^{\alpha}(1)=\left[\ln\left(\frac{2\sigma}{\kappa+\sigma}\right)\right]^{\alpha}+\left[\ln\left(\frac{\kappa+\sigma}{2\kappa}\right)\right]^{\alpha},$$

$$\Delta_{\ln}^{\beta}(1)=\left[\ln\left(\frac{2\varsigma}{\varphi+\varsigma}\right)\right]^{\beta}+\left[\ln\left(\frac{\varphi+\varsigma}{2\varphi}\right)\right]^{\beta}.$$

For power functions, where $g(t)=\frac{t^{\rho}}{\rho}$ and $w(s)=\frac{s^{\sigma}}{\sigma}$, we define

$$\Lambda_{\rho}^{\alpha}(t)=\frac{1}{\rho^{\alpha}}\left[\left(\sigma^{\rho}-\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma\right)^{\rho}\right)^{\alpha}+\left(\left(\frac{2-t}{2}\kappa+\frac{t}{2}\sigma\right)^{\rho}-\kappa^{\rho}\right)^{\alpha}\right],$$

$$\Delta_{\sigma}^{\beta}(s)=\frac{1}{\sigma^{\beta}}\left[\left(\sigma^{\sigma}-\left(\frac{s}{2}\kappa+\frac{2-s}{2}\sigma\right)^{\sigma}\right)^{\beta}+\left(\left(\frac{2-s}{2}\kappa+\frac{s}{2}\sigma\right)^{\sigma}-\kappa^{\sigma}\right)^{\beta}\right].$$

If we take $t=1$ and $s=1$, we get

$$\Lambda_{\rho}^{\alpha}(1)=\frac{1}{\rho^{\alpha}}\left[\left(\sigma^{\rho}-\left(\frac{\kappa+\sigma}{2}\right)^{\rho}\right)^{\alpha}+\left(\left(\frac{\kappa+\sigma}{2}\right)^{\rho}-\kappa^{\rho}\right)^{\alpha}\right],$$

$$\Delta_{\sigma}^{\beta}(1)=\frac{1}{\sigma^{\beta}}\left[\left(\varsigma^{\sigma}-\left(\frac{\varphi+\varsigma}{2}\right)^{\sigma}\right)^{\beta}+\left[\left(\frac{\varphi+\varsigma}{2}\right)^{\sigma}-\varphi^{\sigma}\right]^{\beta}\right].$$

The definitions provided above will be utilized throughout the paper and will play a fundamental role in the derivation of integral inequalities in the subsequent sections. Motivated by all these considerations, by considering co-ordinated convex functions in the setting of generalized fractional integral operators, a more general framework is employed, which includes several well-known fractional integrals such as the Riemann–Liouville, Hadamard, and Katugampola operators. Within this framework, some known results can be recovered as special cases, while several new inequalities are also obtained.

2. Main Results

In this section, we establish new Hermite-Hadamard-type inequalities for co-ordinated convex functions involving generalized fractional integrals. These results extend classical Hermite-Hadamard inequalities by incorporating functions defined on rectangular regions with respect to auxiliary functions g and w.

Theorem 2.1. *If $F:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ, then for $\alpha,\beta>0$, the following Hermite-Hadamard-type inequality holds*

$$\begin{aligned}F\left(\frac{\kappa+\sigma}{2},\frac{\varphi+\varsigma}{2}\right)&\leq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\Lambda_{g}^{\alpha}(1)\Delta_{w}^{\beta}(1)}\left[\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}F(\sigma,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-;g,w}F(\sigma,\varphi)\right.\\&\left.+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+;g,w}F(\kappa,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-;g,w}F(\kappa,\varphi)\right]\\&\leq\frac{F(\kappa,\varphi)+F(\kappa,\varsigma)+F(\sigma,\varphi)+F(\sigma,\varsigma)}{4},\end{aligned}\tag{2.1}$$

where the function F is defined as in (1.7).

Proof. Since F is a co-ordinated convex function on Δ, it satisfies the following inequality

$$F\left(\frac{u+v}{2},\frac{p+q}{2}\right)\leq\frac{F(u,p)+F(u,q)+F(v,p)+F(v,q)}{4}, \tag{2.2}$$

for all $(u,p),(v,q)\in\Delta$. Now, for $t,s\in[0,1]$, let us define

$$u=\frac{2-t}{2}\kappa+\frac{t}{2}\sigma, \qquad v=\frac{t}{2}\kappa+\frac{2-t}{2}\sigma,$$
$$p=\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma, \qquad q=\frac{2-s}{2}\varphi+\frac{s}{2}\varsigma.$$

Substituting these expressions into (2.2), we obtain

$$\begin{aligned}F\left(\frac{\kappa+\sigma}{2},\frac{\varphi+\varsigma}{2}\right)\leq\frac{1}{4}\Bigg[&F\left(\frac{2-t}{2}\kappa+\frac{t}{2}\sigma,\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma\right)+F\left(\frac{2-t}{2}\kappa+\frac{t}{2}\sigma,\frac{2-s}{2}\varphi+\frac{s}{2}\varsigma\right)\\&+F\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma,\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma\right)+F\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma,\frac{2-s}{2}\varphi+\frac{s}{2}\varsigma\right)\Bigg].\end{aligned} \tag{2.3}$$

Multiplying both sides of (2.3) by

$$\frac{(\sigma-\kappa)(\varsigma-\varphi)}{\Gamma(\alpha)\Gamma(\beta)}\frac{g'(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma)}{\left[g(\sigma)-g(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma)\right]^{1-\alpha}}\frac{w'(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma)}{\left[w(\varsigma)-w(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma)\right]^{1-\beta}},$$

and integrating both sides over the region $(t,s)\in(0,1)\times(0,1)$, we obtain

$$\begin{aligned}&\frac{(\sigma-\kappa)(\varsigma-\varphi)}{\Gamma(\alpha)\Gamma(\beta)}F\left(\frac{\kappa+\sigma}{2},\frac{\varphi+\varsigma}{2}\right)\int_0^1\int_0^1\frac{g'(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma)}{\left[g(\sigma)-g(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma)\right]^{1-\alpha}}\\&\quad\times\frac{w'(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma)}{\left[w(\varsigma)-w(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma)\right]^{1-\beta}}\varsigma s\varsigma t\\&\leq\frac{(\sigma-\kappa)(\varsigma-\varphi)}{4\Gamma(\alpha)\Gamma(\beta)}\int_0^1\int_0^1\frac{g'(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma)}{\left[g(\sigma)-g(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma)\right]^{1-\alpha}}\\&\quad\times\frac{w'(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma)}{\left[w(\varsigma)-w(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma)\right]^{1-\beta}}F\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma,\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma\right)\varsigma s\varsigma t,\\&+\frac{(\sigma-\kappa)(\varsigma-\varphi)}{4\Gamma(\alpha)\Gamma(\beta)}\int_0^1\int_0^1\frac{g'(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma)}{\left[g(\sigma)-g(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma)\right]^{1-\alpha}}\\&\quad\times\frac{w'(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma)}{\left[w(\varsigma)-w(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma)\right]^{1-\beta}}F\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma,\frac{2-s}{2}\varphi+\frac{s}{2}\varsigma\right)\varsigma s\varsigma t,\\&+\frac{(\sigma-\kappa)(\varsigma-\varphi)}{4\Gamma(\alpha)\Gamma(\beta)}\int_0^1\int_0^1\frac{g'(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma)}{\left[g(\sigma)-g(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma)\right]^{1-\alpha}}\\&\quad\times\frac{w'(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma)}{\left[w(\varsigma)-w(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma)\right]^{1-\beta}}F\left(\frac{2-t}{2}\kappa+\frac{t}{2}\sigma,\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma\right)\varsigma s\varsigma t,\\&+\frac{(\sigma-\kappa)(\varsigma-\varphi)}{4\Gamma(\alpha)\Gamma(\beta)}\int_0^1\int_0^1\frac{g'(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma)}{\left[g(\sigma)-g(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma)\right]^{1-\alpha}}\end{aligned}$$

$$\times \frac{w'(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma)}{\left[w(\varsigma)-w(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma)\right]^{1-\beta}}\digamma\left(\frac{2-t}{2}\kappa+\frac{t}{2}\sigma,\frac{2-s}{2}\varphi+\frac{s}{2}\varsigma\right)\varsigma s\varsigma t.$$

By a simple calculation, we acquire

$$\int_0^1\int_0^1 \frac{g'\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma\right)}{\left[g(\sigma)-g\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma\right)\right]^{1-\alpha}}\frac{w'\left(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma\right)}{\left[w(\varsigma)-w\left(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma\right)\right]^{1-\beta}}\varsigma s\varsigma t$$
$$=\frac{\left[g(\sigma)-g\left(\frac{\kappa+\sigma}{2}\right)\right]^{\alpha}\left[w(\varsigma)-w\left(\frac{\varphi+\varsigma}{2}\right)\right]^{\beta}}{\alpha\beta(\sigma-\kappa)(\varsigma-\varphi)}.$$

Applying the change of variables

$$\tau=\frac{t}{2}\kappa+\frac{2-t}{2}\sigma,\quad \eta=\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma,$$

the integral inequality can be rewritten as

$$\frac{\left[g(\sigma)-g\left(\frac{\kappa+\sigma}{2}\right)\right]^{\alpha}\left[w(\varsigma)-w\left(\frac{\varphi+\varsigma}{2}\right)\right]^{\beta}}{\alpha\beta(\sigma-\kappa)(\varsigma-\varphi)}\digamma\left(\frac{\kappa+\sigma}{2},\frac{\varphi+\varsigma}{2}\right)$$
$$\leq\frac{1}{4\Gamma(\alpha)\Gamma(\beta)}\int_{\frac{\kappa+\sigma}{2}}^{\sigma}\int_{\frac{\varphi+\varsigma}{2}}^{\varsigma}\frac{g'(\tau)}{[g(\sigma)-g(\tau)]^{1-\alpha}}\frac{w'(\eta)}{[w(\varsigma)-w(\eta)]^{1-\beta}}\digamma(\tau,\eta)\varsigma\eta\varsigma\tau$$
$$+\frac{1}{4\Gamma(\alpha)\Gamma(\beta)}\int_{\frac{\kappa+\sigma}{2}}^{\sigma}\int_{\frac{\varphi+\varsigma}{2}}^{\varsigma}\frac{g'(\tau)}{[g(\sigma)-g(\tau)]^{1-\alpha}}\frac{w'(\eta)}{[w(\varsigma)-w(\eta)]^{1-\beta}}\digamma(\tau,\varphi+\varsigma-\eta)\varsigma\eta\varsigma\tau$$
$$+\frac{1}{4\Gamma(\alpha)\Gamma(\beta)}\int_{\frac{\kappa+\sigma}{2}}^{\sigma}\int_{\frac{\varphi+\varsigma}{2}}^{\varsigma}\frac{g'(\tau)}{[g(\sigma)-g(\tau)]^{1-\alpha}}\frac{w'(\eta)}{[w(\varsigma)-w(\eta)]^{1-\beta}}\digamma(\kappa+\sigma-\tau,\eta)\varsigma\eta\varsigma\tau$$
$$+\frac{1}{4\Gamma(\alpha)\Gamma(\beta)}\int_{\frac{\kappa+\sigma}{2}}^{\sigma}\int_{\frac{\varphi+\varsigma}{2}}^{\varsigma}\frac{g'(\tau)}{[g(\sigma)-g(\tau)]^{1-\alpha}}\frac{w'(\eta)}{[w(\varsigma)-w(\eta)]^{1-\beta}}\digamma(\kappa+\sigma-\tau,\varphi+\varsigma-\eta)\varsigma\eta\varsigma\tau.$$

Rewriting in terms of fractional integrals, we obtain

$$\frac{\left[g(\sigma)-g\left(\frac{\kappa+\sigma}{2}\right)\right]^{\alpha}\left[w(\varsigma)-w\left(\frac{\varphi+\varsigma}{2}\right)\right]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\digamma\left(\frac{\kappa+\sigma}{2},\frac{\varphi+\varsigma}{2}\right)$$
$$\leq\frac{1}{4}\Bigg[\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}\widetilde{\digamma_3}(\sigma,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}\widetilde{\digamma_1}(\sigma,\varsigma)$$
$$+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}\widetilde{\digamma_2}(\sigma,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}\digamma(\sigma,\varsigma)\Bigg].$$

Using the definition of $\digamma(\varpi,\psi)$, the result simplifies to

$$\frac{\left[g(\sigma)-g\left(\frac{\kappa+\sigma}{2}\right)\right]^{\alpha}\left[w(\varsigma)-w\left(\frac{\varphi+\varsigma}{2}\right)\right]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\digamma\left(\frac{\kappa+\sigma}{2},\frac{\varphi+\varsigma}{2}\right)\leq\frac{1}{4}\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}\digamma(\sigma,\varsigma).$$

Similarly, multiplying both sides of (2.3) by

$$\frac{(\sigma-\kappa)(\varsigma-\varphi)}{\Gamma(\alpha)\Gamma(\beta)}\frac{g'\left(\frac{2-t}{2}\kappa+\frac{t}{2}\sigma\right)}{\left[g\left(\frac{2-t}{2}\kappa+\frac{t}{2}\sigma\right)-g(\kappa)\right]^{1-\alpha}}\frac{w'\left(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma\right)}{\left[w(\varsigma)-w\left(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma\right)\right]^{1-\beta}},$$

and integrating the obtained inequality with respect to t,s over $(0,1)\times(0,1)$, we obtain

$$\frac{\left[g\left(\frac{\kappa+\sigma}{2}\right)-g(\kappa)\right]^{\alpha}\left[w(\varsigma)-w\left(\frac{\varphi+\varsigma}{2}\right)\right]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\digamma\left(\frac{\kappa+\sigma}{2},\frac{\varphi+\varsigma}{2}\right)\leq\frac{1}{4}\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+;g,w}\digamma(\kappa,\varsigma).\tag{2.4}$$

Furthermore, multiplying both sides of (2.3) by

$$\frac{(\sigma-\kappa)(\varsigma-\varphi)}{\Gamma(\alpha)\Gamma(\beta)}\frac{g'\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma\right)}{\left[g(\sigma)-g\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma\right)\right]^{1-\alpha}}\frac{w'\left(\frac{2-s}{2}\varphi+\frac{s}{2}\varsigma\right)}{\left[w\left(\frac{2-s}{2}\varphi+\frac{s}{2}\varsigma\right)-w(\varphi)\right]^{1-\beta}},$$

as well as

$$\frac{(\sigma-\kappa)(\varsigma-\varphi)}{\Gamma(\alpha)\Gamma(\beta)}\frac{g'\left(\frac{2-t}{2}\kappa+\frac{t}{2}\sigma\right)}{\left[g\left(\frac{2-t}{2}\kappa+\frac{t}{2}\sigma\right)-g(\kappa)\right]^{1-\alpha}}\frac{w'\left(\frac{2-s}{2}\varphi+\frac{s}{2}\varsigma\right)}{\left[w\left(\frac{2-s}{2}\varphi+\frac{s}{2}\varsigma\right)-w(\varphi)\right]^{1-\beta}},$$

and integrating both expressions over the region $(0,1)\times(0,1)$, we derive the following inequalities

$$\frac{\left[g(\sigma)-g(\frac{\kappa+\sigma}{2})\right]^{\alpha}\left[w\left(\frac{\varphi+\varsigma}{2}\right)-w(\varphi)\right]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}F\left(\frac{\kappa+\sigma}{2},\frac{\varphi+\varsigma}{2}\right)\leq\frac{1}{4}\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-;g,w}F(\sigma,\varphi),\tag{2.5}$$

and

$$\frac{\left[g\left(\frac{\kappa+\sigma}{2}\right)-g(\kappa)\right]^{\alpha}\left[w\left(\frac{\varphi+\varsigma}{2}\right)-w(\varphi)\right]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}F\left(\frac{\kappa+\sigma}{2},\frac{\varphi+\varsigma}{2}\right)\leq\frac{1}{4}\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-;g,w}F(\kappa,\varphi).\tag{2.6}$$

Summing the inequalities (2)-(2.6), we obtain

$$\begin{aligned}F\left(\frac{\kappa+\sigma}{2},\frac{\varphi+\varsigma}{2}\right)&\leq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\Lambda^{\alpha}_{g}(1)\Delta^{\beta}_{w}(1)}\\&\quad\times\left[\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}F(\sigma,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-;g,w}F(\sigma,\varphi)\right.\\&\quad\left.+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+;g,w}F(\kappa,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-;g,w}F(\kappa,\varphi)\right].\end{aligned}$$

This concludes the proof of the first inequality in (2.1).

To prove the second inequality in (2.1), we utilize the fact that F is a co-ordinated convex function, which implies

$$\begin{aligned}&F\left(\frac{2-t}{2}\kappa+\frac{t}{2}\sigma,\frac{2-s}{2}\varphi+\frac{s}{2}\varsigma\right)+F\left(\frac{2-t}{2}\kappa+\frac{t}{2}\sigma,\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma\right)\\&\quad+F\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma,\frac{2-s}{2}\varphi+\frac{s}{2}\varsigma\right)+F\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma,\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma\right)\\&\leq F(\kappa,\varphi)+F(\kappa,\varsigma)+F(\sigma,\varphi)+F(\sigma,\varsigma).\end{aligned}\tag{2.7}$$

Multiplying both sides of (2.7) by

$$\frac{(\sigma-\kappa)(\varsigma-\varphi)}{\Gamma(\alpha)\Gamma(\beta)}\frac{g'\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma\right)}{\left[g(\sigma)-g\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma\right)\right]^{1-\alpha}}\frac{w'\left(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma\right)}{\left[w(\varsigma)-w\left(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma\right)\right]^{1-\beta}},$$

and integrating over $(0,1)\times(0,1)$, we obtain

$$\begin{aligned}&\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}\widetilde{F}_3(\sigma,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}\widetilde{F}_1(\sigma,\varsigma)\\&\quad+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}\widetilde{F}_2(\sigma,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}F(\sigma,\varsigma)\\&\leq\left[F(\kappa,\varphi)+F(\kappa,\varsigma)+F(\sigma,\varphi)+F(\sigma,\varsigma)\right]\frac{\left[g(\sigma)-g\left(\frac{\kappa+\sigma}{2}\right)\right]^{\alpha}\left[w(\varsigma)-w\left(\frac{\varphi+\varsigma}{2}\right)\right]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}.\end{aligned}$$

Thus, we have

$$\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}F(\sigma,\varsigma)\leq\left[F(\kappa,\varphi)+F(\kappa,\varsigma)+F(\sigma,\varphi)+F(\sigma,\varsigma)\right]\frac{\left[g(\sigma)-g\left(\frac{\kappa+\sigma}{2}\right)\right]^{\alpha}\left[w(\varsigma)-w\left(\frac{\varphi+\varsigma}{2}\right)\right]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}.\tag{2.8}$$

Similarly, multiplying both sides of (2.7) by

$$\frac{(\sigma-\kappa)(\varsigma-\varphi)}{\Gamma(\alpha)\Gamma(\beta)}\frac{g'\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma\right)}{\left[g(\sigma)-g\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma\right)\right]^{1-\alpha}}\frac{w'\left(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma\right)}{\left[w\left(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma\right)-w(\varphi)\right]^{1-\beta}},$$

$$\frac{(\sigma-\kappa)(\varsigma-\varphi)}{\Gamma(\alpha)\Gamma(\beta)}\frac{g'\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma\right)}{\left[g\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma\right)-g(\kappa)\right]^{1-\alpha}}\frac{w'\left(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma\right)}{\left[w(\varsigma)-w\left(\frac{s}{2}\varphi+\frac{2-s}{2}\varsigma\right)\right]^{1-\beta}},$$

and

$$\frac{(\sigma-\kappa)(\varsigma-\varphi)}{\Gamma(\alpha)\Gamma(\beta)}\frac{g'\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma\right)}{\left[g\left(\frac{t}{2}\kappa+\frac{2-t}{2}\sigma\right)-g(\kappa)\right]^{1-\alpha}}\frac{w'\left(\frac{s}{2}\kappa+\frac{2-s}{2}\sigma\right)}{\left[w\left(\frac{s}{2}\kappa+\frac{2-s}{2}\sigma\right)-w(\varphi)\right]^{1-\beta}},$$

and integrating the resulting inequalities over $(0,1)\times(0,1)$, we establish the following inequalities

$$\begin{aligned}\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-;g,w}F(\sigma,\varphi) &\leq [F(\kappa,\varphi)+F(\kappa,\varsigma)+F(\sigma,\varphi)+F(\sigma,\varsigma)]\\ &\quad\times\frac{\left[g(\sigma)-g\left(\frac{\kappa+\sigma}{2}\right)\right]^{\alpha}\left[w(\frac{\varphi+\varsigma}{2})-w(\varphi)\right]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}.\end{aligned}\tag{2.9}$$

$$\begin{aligned}\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+;g,w}F(\kappa,\varsigma) &\leq [F(\kappa,\varphi)+F(\kappa,\varsigma)+F(\sigma,\varphi)+F(\sigma,\varsigma)]\\ &\quad\times\frac{\left[g(\frac{\kappa+\sigma}{2})-g(\kappa)\right]^{\alpha}\left[w(\varsigma)-w\left(\frac{\varphi+\varsigma}{2}\right)\right]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}.\end{aligned}\tag{2.10}$$

$$\begin{aligned}\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-;g,w}F(\kappa,\varphi) &\leq [F(\kappa,\varphi)+F(\kappa,\varsigma)+F(\sigma,\varphi)+F(\sigma,\varsigma)]\\ &\quad\times\frac{\left[g(\frac{\kappa+\sigma}{2})-g(\kappa)\right]^{\alpha}\left[w(\frac{\varphi+\varsigma}{2})-w(\varphi)\right]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)},\end{aligned}\tag{2.11}$$

respectively. By summing the inequalities (2.8)-(2.11), we get

$$\begin{aligned}&\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}F(\sigma,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-;g,w}F(\sigma,\varphi)\\ &\quad+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+;g,w}F(\kappa,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-;g,w}F(\kappa,\varphi)\\ &\leq [F(\kappa,\varphi)+F(\kappa,\varsigma)+F(\sigma,\varphi)+F(\sigma,\varsigma)]\,\frac{\Lambda^{\alpha}_{g}(1)\Delta^{\beta}_{w}(1)}{4\Gamma(\alpha+1)\Gamma(\beta+1)}.\end{aligned}\tag{2.12}$$

Thus, the second inequality in (2.1) is established. This concludes the proof. □

Corollary 2.1. *If we set $g(t)=t$ and $w(s)=s$ for Theorem 2.1, we obtain the following inequalities for the Riemann-Liouville fractional integrals*

$$\begin{aligned}F\left(\frac{\kappa+\sigma}{2},\frac{\varphi+\varsigma}{2}\right) &\leq \frac{2^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\sigma-\kappa)^{\alpha}(\varsigma-\varphi)^{\beta}}\\ &\quad\times\left[J^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+}F(\sigma,\varsigma)+J^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-}F(\sigma,\varphi)\right.\\ &\quad\left.+J^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+}F(\kappa,\varsigma)+J^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-}F(\kappa,\varphi)\right]\\ &\leq \frac{F(\kappa,\varphi)+F(\kappa,\varsigma)+F(\sigma,\varphi)+F(\sigma,\varsigma)}{4}.\end{aligned}\tag{2.13}$$

Example 2.1. Let $F:[1,3]\times[1,3]\to\mathbb{R}$ be the function defined by

$$F(t,s)=t^3s^3.$$

We now verify the inequality given in Corollary 2.13 for this function. Evaluating F at the midpoint of the region, we obtain

$$F\left(\frac{\kappa+\sigma}{2},\frac{\varphi+\varsigma}{2}\right)=F(2,2)=64.$$

Next, we compute the fractional integral term in (2.13). Substituting the given function, this term can be expressed as

$$\begin{aligned}
&\frac{2^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\sigma-\kappa)^{\alpha}(\varsigma-\varphi)^{\beta}}\\
&\times\left[J^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+}F(\sigma,\varsigma)+J^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-}F(\sigma,\varphi)+J^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+}F(\kappa,\varsigma)+J^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-}F(\kappa,\varphi)\right]\\
&=\frac{\alpha\beta}{4}\left[\int_2^3\int_2^3(3-t)^{\alpha-1}(3-s)^{\beta-1}t^3s^3\varsigma s\varsigma t+\int_2^3\int_1^2(3-t)^{\alpha-1}(s-1)^{\beta-1}t^3s^3\varsigma s\varsigma t\right.\\
&\quad\left.+\int_1^2\int_2^3(t-1)^{\alpha-1}(3-s)^{\beta-1}t^3s^3\varsigma s\varsigma t+\int_1^2\int_1^2(t-1)^{\alpha-1}(s-1)^{\beta-1}t^3s^3\varsigma s\varsigma t\right].\\
&=16\left[\frac{2\alpha^2+6\alpha+7}{(\alpha+1)(\alpha+2)}\right]\left[\frac{2\beta^2+6\beta+7}{(\beta+1)(\beta+2)}\right].
\end{aligned}$$

Finally, the right-hand side of (2.13) is given by the arithmetic mean of the function values at the four corner points:

$$\begin{aligned}
\frac{F(\kappa,\varphi)+F(\kappa,\varsigma)+F(\sigma,\varphi)+F(\sigma,\varsigma)}{4}&=\frac{F(1,1)+F(1,3)+F(3,1)+F(3,3)}{4}\\
&=\frac{1+27+27+729}{4}=196.
\end{aligned}$$

Consequently, we have

$$64<16\left[\frac{2\alpha^2+6\alpha+7}{(\alpha+1)(\alpha+2)}\right]\left[\frac{2\beta^2+6\beta+7}{(\beta+1)(\beta+2)}\right]<196 \tag{2.14}$$

Thus, the inequality (2.14) holds for this function. Figure 1 illustrates the validity of Example 2.1, where the parameters α and β are considered within the interval $[0,6]$.

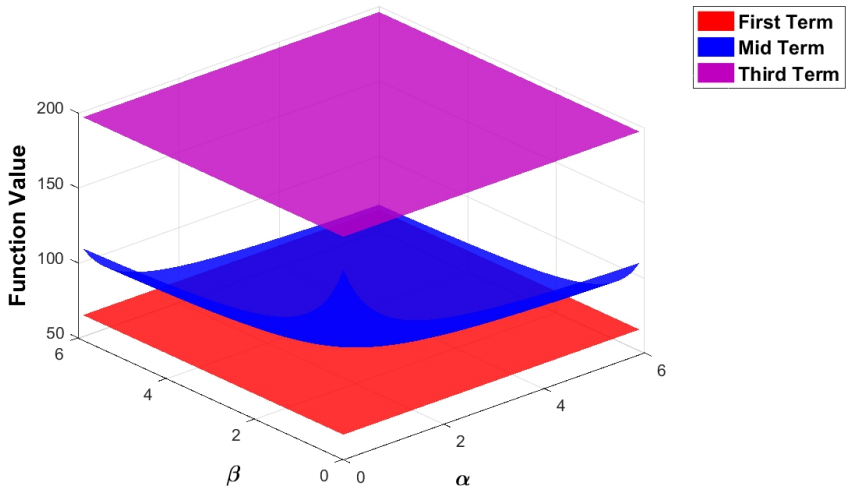

Figure 1. The graph of Example 2.1 has been generated using MATLAB, illustrating its dependence on the parameters α and β, where $\alpha,\beta\in[0,6]$.

Remark 2.1. If we set $\alpha = 1$ and $\beta = 1$ for Theorem 2.1, we obtain the classical double integral form of Hermite-Hadamard-type inequalities

$$F\left(\frac{\kappa+\sigma}{2}, \frac{\varphi+\varsigma}{2}\right) \leq \frac{1}{(\sigma-\kappa)(\varsigma-\varphi)} \int_{\kappa}^{\sigma}\int_{\varphi}^{\varsigma} F(t,s)\varsigma s\varsigma t \leq \frac{F(\kappa,\varphi)+F(\kappa,\varsigma)+F(\sigma,\varphi)+F(\sigma,\varsigma)}{4},$$

which corresponds to the result established by Dragomir in [13].

Corollary 2.2. *Under the assumptions of Theorem 2.1, if we choose $g(t) = \ln t$ and $w(s) = \ln s$, we obtain the following inequalities for Hadamard fractional integrals*

$$\begin{aligned} F\left(\frac{\kappa+\sigma}{2}, \frac{\varphi+\varsigma}{2}\right) &\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\Lambda_{\ln}^{\alpha}(1)\Delta_{\ln}^{\beta}(1)} \\ &\quad \times \left[\mathbf{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+}F(\sigma,\varsigma) + \mathbf{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-}F(\sigma,\varphi)\right. \\ &\quad \left. + \mathbf{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+}F(\kappa,\varsigma) + \mathbf{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-}F(\kappa,\varphi)\right] \\ &\leq \frac{F(\kappa,\varphi)+F(\kappa,\varsigma)+F(\sigma,\varphi)+F(\sigma,\varsigma)}{4}. \end{aligned} \tag{2.15}$$

Corollary 2.3. *Under the assumptions of Theorem 2.1, if we set $g(t) = \frac{t^{\rho}}{\rho}$ and $w(s) = \frac{s^{\sigma}}{\sigma}$, then we obtain the following inequalities for Katugampola fractional integrals:*

$$\begin{aligned} F\left(\frac{\kappa+\sigma}{2}, \frac{\varphi+\varsigma}{2}\right) &\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\Lambda_{\rho}^{\alpha}(1)\Delta_{\sigma}^{\beta}(1)} \\ &\quad \times \left[{}^{\rho,\sigma}\mathcal{I}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}F(\sigma,\varsigma) + {}^{\rho,\sigma}\mathcal{I}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-;g,w}F(\sigma,\varphi)\right. \\ &\quad \left. + {}^{\rho,\sigma}\mathcal{I}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+;g,w}F(\kappa,\varsigma) + {}^{\rho,\sigma}\mathcal{I}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-;g,w}F(\kappa,\varphi)\right] \\ &\leq \frac{F(\kappa,\varphi)+F(\kappa,\varsigma)+F(\sigma,\varphi)+F(\sigma,\varsigma)}{4}. \end{aligned}$$

Theorem 2.2. *If the function $F : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ, then for $\alpha, \beta > 0$, the following Hermite-Hadamard type inequality holds*

$$\begin{aligned} F\left(\frac{\kappa+\sigma}{2}, \frac{\varphi+\varsigma}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{4\Lambda_{g}^{\alpha}(1)}\left[\mathcal{J}^{\alpha}_{\frac{\kappa+\sigma}{2}+;g}H\left(\sigma, \frac{\varphi+\varsigma}{2}\right) + \mathcal{J}^{\alpha}_{\frac{\kappa+\sigma}{2}-;g}H\left(\kappa, \frac{\varphi+\varsigma}{2}\right)\right] \\ &\quad + \frac{\Gamma(\beta+1)}{4\Delta_{w}^{\beta}(1)}\left[\mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}+;w}G\left(\frac{\kappa+\sigma}{2}, \varsigma\right) + \mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}-;w}G\left(\frac{\kappa+\sigma}{2}, \varphi\right)\right] \\ &\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\Lambda_{g}^{\alpha}(1)\Delta_{w}^{\beta}(1)} \\ &\quad \times \left[\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}F(\sigma,\varsigma) + \mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-;g,w}F(\sigma,\varphi)\right. \\ &\quad \left. + \mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+;g,w}F(\kappa,\varsigma) + \mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-;g,w}F(\kappa,\varphi)\right] \\ &\leq \frac{\Gamma(\alpha+1)}{8\Lambda_{g}^{\alpha}(1)}\left[\mathcal{J}^{\alpha}_{\frac{\kappa+\sigma}{2}+;g}H(\sigma,\varphi) + \mathcal{J}^{\alpha}_{\frac{\kappa+\sigma}{2}+;g}H(\sigma,\varsigma)\right. \\ &\quad \left. + \mathcal{J}^{\alpha}_{\frac{\kappa+\sigma}{2}-;g}H(\kappa,\varphi) + \mathcal{J}^{\alpha}_{\frac{\kappa+\sigma}{2}-;g}H(\kappa,\varsigma)\right] \\ &\quad + \frac{\Gamma(\beta+1)}{8\Delta_{w}^{\beta}(1)}\left[\mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}+;w}G(\kappa,\varsigma) + \mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}+;w}G(\sigma,\varsigma)\right. \\ &\quad \left. + \mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}-;w}G(\kappa,\varphi) + \mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}-;w}G(\sigma,\varphi)\right] \end{aligned}$$

$$\leq \frac{\digamma(\kappa,\varphi)+\digamma(\kappa,\varsigma)+\digamma(\sigma,\varphi)+\digamma(\sigma,\varsigma)}{4}, \tag{2.16}$$

where the functions H, $\digamma$, and G are defined as in (1.7).

Proof. Let us define the function $h_{\psi}^{1}:[\kappa,\sigma]\to\mathbb{R}$ by

$$h_{\psi}^{1}(\varpi)=\digamma(\varpi,\psi).$$

Since $\digamma$ is co-ordinated convex on Δ, it follows that $h_{\psi}^{1}(\varpi)$ is convex on $[\kappa,\sigma]$ for all $\psi\in[\varphi,\varsigma]$. Furthermore, defining

$$H_{\psi}^{1}(\varpi)=h_{\psi}^{1}(\varpi)+\widetilde{h_{\psi}^{1}}(\varpi)=\digamma(\varpi,\psi)+\widetilde{F_{1}}(\varpi,\psi)=H(\varpi,\psi),$$

and applying the inequality (1.5) to the convex function $h_{\psi}^{1}(\varpi)$, we obtain

$$\begin{aligned} h_{\psi}^{1}\left(\frac{\kappa+\sigma}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2\Lambda_{g}^{\alpha}(1)}\left[\mathcal{J}_{\frac{\kappa+\sigma}{2}+;g}^{\alpha}H_{\psi}^{1}(\sigma)+\mathcal{J}_{\frac{\kappa+\sigma}{2}-;w}^{\alpha}H_{\psi}^{1}(\kappa)\right] \\ &\leq \frac{h_{\psi}^{1}(\kappa)+h_{\psi}^{1}(\sigma)}{2}. \end{aligned}$$

This directly implies that

$$\begin{aligned} \digamma\left(\frac{\kappa+\sigma}{2},\psi\right) &\leq \frac{\alpha}{2\Lambda_{g}^{\alpha}(1)}\left[\int_{\frac{\kappa+\sigma}{2}}^{\sigma}\frac{g'(\varpi)}{[g(\sigma)-g(\varpi)]^{1-\alpha}}H(\varpi,\psi)\varsigma\varpi+\int_{\kappa}^{\frac{\kappa+\sigma}{2}}\frac{g'(\varpi)}{[g(\varpi)-g(\kappa)]^{1-\alpha}}H(\varpi,\psi)\varsigma\varpi\right] \\ &\leq \frac{\digamma(\kappa,\psi)+\digamma(\sigma,\psi)}{2}. \end{aligned} \tag{2.17}$$

Next, multiplying both sides of inequality (2.17) by

$$\frac{\beta}{2\Delta_{w}^{\beta}(1)}\frac{w'(\psi)}{[w(\varsigma)-w(\psi)]^{1-\beta}},$$

and integrating over $\left[\frac{\varphi+\varsigma}{2},\varsigma\right]$, we acquire

$$\begin{aligned} \frac{\Gamma(\beta+1)}{2\Delta_{w}^{\beta}(1)}\mathcal{J}_{\frac{\varphi+\varsigma}{2}+;w}^{\beta}\digamma\left(\frac{\kappa+\sigma}{2},\varsigma\right) &\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\Lambda_{g}^{\alpha}(1)\Delta_{w}^{\beta}(1)} \\ &\quad\times\left[\mathcal{J}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}^{\alpha,\beta}H(\sigma,\varsigma)+\mathcal{J}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+;g,w}^{\alpha,\beta}H(\kappa,\varsigma)\right] \\ &\leq \frac{\Gamma(\beta+1)}{2\Delta_{w}^{\beta}(1)}\left[\mathcal{J}_{\frac{\varphi+\varsigma}{2}+;w}^{\beta}\digamma(\kappa,\varsigma)+\mathcal{J}_{\frac{\varphi+\varsigma}{2}+;w}^{\beta}\digamma(\sigma,\varsigma)\right]. \end{aligned} \tag{2.18}$$

Similarly, multiplying (2.17) by

$$\frac{\beta}{2\Delta_{w}^{\beta}(1)}\frac{w'(\psi)}{[w(\psi)-w(\varphi)]^{1-\beta}},$$

and integrating over $\left[\varphi,\frac{\varphi+\varsigma}{2}\right]$, we establish

$$\begin{aligned} \frac{\Gamma(\beta+1)}{2\Delta_{w}^{\beta}(1)}\mathcal{J}_{\frac{\varphi+\varsigma}{2}-;w}^{\beta}\digamma\left(\frac{\kappa+\sigma}{2},\varphi\right) &\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\Lambda_{g}^{\alpha}(1)\Delta_{w}^{\beta}(1)} \\ &\quad\times\left[\mathcal{J}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-;g,w}^{\alpha,\beta}H(\sigma,\varphi)+\mathcal{J}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-;g,w}^{\alpha,\beta}H(\kappa,\varphi)\right] \\ &\leq \frac{\Gamma(\beta+1)}{2\Delta_{w}^{\beta}(1)}\left[\mathcal{J}_{\frac{\varphi+\varsigma}{2}-;w}^{\beta}\digamma(\kappa,\varphi)+\mathcal{J}_{\frac{\varphi+\varsigma}{2}-;w}^{\beta}\digamma(\sigma,\varphi)\right]. \end{aligned} \tag{2.19}$$

Furthermore, let us define the function $h_{\psi}^{2}:[\kappa,\sigma]\to\mathbb{R}$ as

$$h_{\psi}^{2}(\varpi)=\widetilde{F_{2}}(\varpi,\psi).$$

Since F is co-ordinated convex, it follows that $h_\psi^2(\varpi)$ is convex on $[\kappa,\sigma]$ for all $\psi\in[\varphi,\varsigma]$. Moreover, defining

$$H_\psi^2(\varpi)=h_\psi^2(\varpi)+\widetilde{h_\psi^2}(\varpi)=\widetilde{F}_2(\varpi,\psi)+\widetilde{F}_3(\varpi,\psi)=L(\varpi,\psi),$$

and applying inequality (1.5) to the convex function $h_\psi^2(\varpi)$, we obtain

$$\begin{aligned}h_\psi^2\left(\frac{\kappa+\sigma}{2}\right)&\le\frac{\Gamma(\alpha+1)}{2\Lambda_g^\alpha(1)}\Big[\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}+;g}H_\psi^2(\sigma)+\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}-;w}H_\psi^2(\kappa)\Big]\\&\le\frac{h_\psi^2(\kappa)+h_\psi^2(\sigma)}{2}.\end{aligned}$$

This directly implies that

$$\begin{aligned}\widetilde{F}_2\left(\frac{\kappa+\sigma}{2},\psi\right)&\le\frac{\alpha}{2\Lambda_g^\alpha(1)}\Bigg[\int\limits_{\frac{\kappa+\sigma}{2}}^{\sigma}\frac{g'(\varpi)}{[g(\sigma)-g(\varpi)]^{1-\alpha}}L(\varpi,\psi)\varsigma\varpi+\int\limits_{\kappa}^{\frac{\kappa+\sigma}{2}}\frac{g'(\varpi)}{[g(\varpi)-g(\kappa)]^{1-\alpha}}L(\varpi,\psi)\varsigma\varpi\Bigg]\\&\le\frac{\widetilde{F}_2(\kappa,\psi)+\widetilde{F}_2(\sigma,\psi)}{2}.\end{aligned}\tag{2.20}$$

Multiplying both sides of inequality (2.20) by

$$\frac{\beta}{2\Delta_w^\beta(1)}\frac{w'(\psi)}{[w(\varsigma)-w(\psi)]^{1-\beta}}$$

and

$$\frac{\beta}{2\Delta_w^\beta(1)}\frac{w'(\psi)}{[w(\psi)-w(\varphi)]^{1-\beta}},$$

then integrating the resulting expressions with respect to ψ over the intervals $\left(\frac{\varphi+\varsigma}{2},\varsigma\right)$ and $\left(\varphi,\frac{\varphi+\varsigma}{2}\right)$, respectively, we obtain the following inequalities

$$\begin{aligned}\frac{\Gamma(\beta+1)}{2\Delta_w^\beta(1)}\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}+;w}\widetilde{F}_2\left(\frac{\kappa+\sigma}{2},\varsigma\right)&\le\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\Lambda_g^\alpha(1)\Delta_w^\beta(1)}\Big[\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}L(\sigma,\varsigma)\\&\quad+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+;g,w}L(\kappa,\varsigma)\Big]\\&\le\frac{\Gamma(\beta+1)}{4\Delta_w^\beta(1)}\Big[\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}+;w}\widetilde{F}_2(\kappa,\varsigma)+\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}+;w}\widetilde{F}_2(\sigma,\varsigma)\Big].\end{aligned}\tag{2.21}$$

Similarly, we also obtain:

$$\begin{aligned}\frac{\Gamma(\beta+1)}{2\Delta_w^\beta(1)}\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}-;w}\widetilde{F}_2\left(\frac{\kappa+\sigma}{2},\varphi\right)&\le\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\Lambda_g^\alpha(1)\Delta_w^\beta(1)}\Big[\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-;g,w}L(\sigma,\varphi)\\&\quad+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-;g,w}L(\kappa,\varphi)\Big]\\&\le\frac{\Gamma(\beta+1)}{4\Delta_w^\beta(1)}\Big[\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}-;w}\widetilde{F}_2(\kappa,\varphi)+\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}-;w}\widetilde{F}_2(\sigma,\varphi)\Big].\end{aligned}\tag{2.22}$$

Since F is co-ordinated convex on Δ, we define the function $h^1_\varpi:[\varphi,\varsigma]\to\mathbb{R}$ as

$$h^1_\varpi(\psi)=F(\varpi,\psi).$$

Since $h^1_\varpi(\psi)$ is convex for all $\varpi\in[\kappa,\sigma]$, we introduce the function

$$H^1_\varpi(\psi)=h^1_\varpi(\psi)+\widetilde{h^1_\varpi}(\psi)=F(\varpi,\psi)+\widetilde{F}_2(\varpi,\psi)=G(\varpi,\psi).$$

Applying inequality (1.5) to the convex function $h^1_\varpi(\psi)$, we obtain

$$h^1_\varpi\left(\frac{\varphi+\varsigma}{2}\right)\le\frac{\Gamma(\beta+1)}{2\Delta_w^\beta(1)}\left[\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}+;w}H^1_\varpi(\varsigma)+\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}-;w}H^1_\varpi(\varphi)\right]\le\frac{h^1_\varpi(\varphi)+h^1_\varpi(\varsigma)}{2}.$$

Rewriting this expression in terms of F, we obtain

$$F\left(\varpi, \frac{\varphi+\varsigma}{2}\right) \leq \frac{\beta}{2\Delta_w^\beta(1)}\left[\int_\varphi^\varsigma \frac{w'(\psi)}{[w(\varsigma)-w(\psi)]^{1-\beta}}G(\varpi,\psi)\varsigma\psi + \int_\varphi^\varsigma \frac{w'(\psi)}{[w(\psi)-w(\varphi)]^{1-\beta}}G(\varpi,\psi)\varsigma\psi\right] \leq \frac{F(\varpi,\varphi)+F(\varpi,\varsigma)}{2}. \tag{2.23}$$

Next, multiplying both sides of inequality (2.23) by

$$\frac{\alpha}{2\Lambda_g^\alpha(1)}\frac{g'(\varpi)}{[g(\sigma)-g(\varpi)]^{1-\alpha}},$$

and

$$\frac{\alpha}{2\Lambda_g^\alpha(1)}\frac{g'(\varpi)}{[g(\varpi)-g(\kappa)]^{1-\alpha}},$$

and integrating the resulting expressions over the intervals $\left(\frac{\kappa+\sigma}{2}, \sigma\right)$ and $\left(\kappa, \frac{\kappa+\sigma}{2}\right)$, respectively, we obtain

$$\begin{aligned}\frac{\Gamma(\alpha+1)}{2\Lambda_g^\alpha(1)}\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}+;g}F\left(\sigma, \frac{\varphi+\varsigma}{2}\right) &\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\Lambda_g^\alpha(1)\Delta_w^\beta(1)}\Big[\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}G(\sigma,\varsigma) \\ &\quad + \mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-;g,w}G(\sigma,\varphi)\Big] \\ &\leq \frac{\Gamma(\alpha+1)}{4\Lambda_g^\alpha(1)}\Big[\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}+;g}F(\sigma,\varphi) + \mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}+;g}F(\sigma,\varsigma)\Big].\end{aligned} \tag{2.24}$$

Similarly, we also obtain:

$$\begin{aligned}\frac{\Gamma(\alpha+1)}{2\Lambda_g^\alpha(1)}\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}-;g}F\left(\kappa, \frac{\varphi+\varsigma}{2}\right) &\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\Lambda_g^\alpha(1)\Delta_w^\beta(1)}\Big[\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+;g,w}G(\kappa,\varsigma) \\ &\quad + \mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-;g,w}G(\kappa,\varphi)\Big] \\ &\leq \frac{\Gamma(\alpha+1)}{4\Lambda_g^\alpha(1)}\Big[\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}-;g}F(\kappa,\varphi) + \mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}-;g}F(\kappa,\varsigma)\Big].\end{aligned} \tag{2.25}$$

On the other hand, since F is co-ordinated convex on Δ, we define the function $h^2_\varpi : [\varphi,\varsigma] \to \mathbb{R}$ as

$$h^2_\varpi(\psi) = \widetilde{F_1}(\varpi,\psi).$$

Since $h^2_\varpi(\psi)$ is convex for all $\varpi \in [\kappa,\sigma]$, we introduce the function

$$H^2_\varpi(\psi) = h^2_\varpi(\psi) + \widetilde{h^2_\varpi}(\psi) = \widetilde{F_1}(\varpi,\psi) + \widetilde{F_3}(\varpi,\psi) = K(\varpi,\psi).$$

Applying inequality (1.5) to the convex function $h^2_\varpi(\psi)$, we obtain

$$h^2_\varpi\left(\frac{\varphi+\varsigma}{2}\right) \leq \frac{\Gamma(\beta+1)}{2\Delta_w^\beta(1)}\left[\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}+;w}H^2_\varpi(\varsigma) + \mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}-;w}H^2_\varpi(\varphi)\right] \leq \frac{h^2_\varpi(\varphi)+h^2_\varpi(\varsigma)}{2}.$$

Rewriting this expression in terms of $\widetilde{F_1}$, we obtain

$$\widetilde{F_1}\left(\varpi, \frac{\varphi+\varsigma}{2}\right) \leq \frac{\beta}{2\Delta_w^\beta(1)}\left[\int_{\frac{\varphi+\varsigma}{2}}^\varsigma \frac{w'(\psi)}{[w(\varsigma)-w(\psi)]^{1-\beta}}K(\varpi,\psi)\varsigma\psi + \int_\varphi^{\frac{\varphi+\varsigma}{2}} \frac{w'(\psi)}{[w(\psi)-w(\varphi)]^{1-\beta}}K(\varpi,\psi)\varsigma\psi\right] \leq \frac{\widetilde{F_1}(\varpi,\varphi)+\widetilde{F_1}(\varpi,\varsigma)}{2}. \tag{2.26}$$

Similarly, multiplying both sides of (2.26) by

$$\frac{\alpha}{2\Lambda_g^\alpha(1)}\frac{g'(\varpi)}{[g(\sigma)-g(\varpi)]^{1-\alpha}},$$

and

$$\frac{\alpha}{2\Lambda_g^\alpha(1)}\frac{g'(\varpi)}{[g(\varpi)-g(\kappa)]^{1-\alpha}},$$

and integrating the resulting inequalities over the intervals $\left[\frac{\kappa+\sigma}{2},\sigma\right]$ and $\left[\kappa,\frac{\kappa+\sigma}{2}\right]$, we obtain the following inequalities

$$\begin{aligned}\frac{\Gamma(\alpha+1)}{2\Lambda_g^\alpha(1)}\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}+;g}\widetilde{F_1}\left(\sigma,\frac{\varphi+\varsigma}{2}\right)&\leq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\Lambda_g^\alpha(1)\Delta_w^\beta(1)}\left[\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}K(\sigma,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-;g,w}K(\sigma,\varphi)\right]\\&\leq\frac{\Gamma(\alpha+1)}{4\Lambda_g^\alpha(1)}\left[\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}+;g}\widetilde{F_1}(\sigma,\varphi)+\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}+;g}\widetilde{F_1}(\sigma,\varsigma)\right],\end{aligned}\tag{2.27}$$

and

$$\begin{aligned}\frac{\Gamma(\alpha+1)}{2\Lambda_g^\alpha(1)}\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}-;g}\widetilde{F_1}\left(\kappa,\frac{\varphi+\varsigma}{2}\right)&\leq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\Lambda_g^\alpha(1)\Delta_w^\beta(1)}\left[\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+;g,w}K(\kappa,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-;g,w}K(\kappa,\varphi)\right]\\&\leq\frac{\Gamma(\alpha+1)}{4\Lambda_g^\alpha(1)}\left[\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}-;g}\widetilde{F_1}(\kappa,\varphi)+\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}-;g}\widetilde{F_1}(\kappa,\varsigma)\right].\end{aligned}\tag{2.28}$$

By summing the inequalities (2.18), (2.19), (2.21), (2.22),(2.24), (2.25), (2.27), and (2.28), we obtain the following relation

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2\Lambda_g^\alpha(1)}\left[\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}+;g}F\left(\sigma,\frac{\varphi+\varsigma}{2}\right)+\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}-;g}F\left(\kappa,\frac{\varphi+\varsigma}{2}\right)\right.\\&\quad\left.+\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}+;g}\widetilde{F_1}\left(\sigma,\frac{\varphi+\varsigma}{2}\right)+\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}-;g}\widetilde{F_1}\left(\kappa,\frac{\varphi+\varsigma}{2}\right)\right]\\&+\frac{\Gamma(\beta+1)}{2\Delta_w^\beta(1)}\left[\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}+;w}F\left(\frac{\kappa+\sigma}{2},\varsigma\right)+\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}-;w}F\left(\frac{\kappa+\sigma}{2},\varphi\right)\right.\\&\quad\left.+\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}+;w}\widetilde{F_2}\left(\frac{\kappa+\sigma}{2},\varsigma\right)+\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}-;w}\widetilde{F_2}\left(\frac{\kappa+\sigma}{2},\varphi\right)\right]\\&\leq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\Lambda_g^\alpha(1)\Delta_w^\beta(1)}\left[\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}G(\sigma,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-;g,w}G(\sigma,\varphi)\right.\\&\quad+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+;g,w}G(\kappa,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-;g,w}G(\kappa,\varphi)\\&\quad+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}K(\sigma,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-;g,w}K(\sigma,\varphi)\\&\quad+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+;g,w}K(\kappa,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-;g,w}K(\kappa,\varphi)\\&\quad+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}H(\sigma,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+;g,w}H(\kappa,\varsigma)\\&\quad+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-;g,w}H(\sigma,\varphi)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-;g,w}H(\kappa,\varphi)\\&\quad+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}L(\sigma,\varsigma)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+;g,w}L(\kappa,\varsigma)\\&\quad\left.+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-;g,w}L(\sigma,\varphi)+\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-;g,w}L(\kappa,\varphi)\right]\\&\leq\frac{\Gamma(\alpha+1)}{4\Lambda_g^\alpha(1)}\left[\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}+;g}F(\sigma,\varphi)+\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}+;g}F(\sigma,\varsigma)\right.\\&\quad+\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}-;g}F(\kappa,\varphi)+\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}-;g}F(\kappa,\varsigma)\\&\quad+\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}+;g}\widetilde{F_1}(\sigma,\varphi)+\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}+;g}\widetilde{F_1}(\sigma,\varsigma)\\&\quad\left.+\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}-;g}\widetilde{F_1}(\kappa,\varphi)+\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}-;g}\widetilde{F_1}(\kappa,\varsigma)\right]\\&+\frac{\Gamma(\beta+1)}{4\Delta_w^\beta(1)}\left[\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}+;w}F(\kappa,\varsigma)+\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}+;w}F(\sigma,\varsigma)\right.\end{aligned}$$

$$
\begin{aligned}
&+ \mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}-;w}F(\kappa,\varphi) + \mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}-;w}F(\sigma,\varphi)\\
&+ \mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}+;w}\widetilde{F}_2(\kappa,\varsigma) + \mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}+;w}\widetilde{F}_2(\sigma,\varsigma)\\
&+ \mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}-;w}\widetilde{F}_2(\kappa,\varphi) + \mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}-;w}\widetilde{F}_2(\sigma,\varphi)\Big].
\end{aligned}
$$

That is, we obtain the following inequality

$$
\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2\Lambda^{\alpha}_{g}(1)}\left[\mathcal{J}^{\alpha}_{\frac{\kappa+\sigma}{2}+;g}H\left(\sigma,\frac{\varphi+\varsigma}{2}\right) + \mathcal{J}^{\alpha}_{\frac{\kappa+\sigma}{2}-;g}H\left(\kappa,\frac{\varphi+\varsigma}{2}\right)\right]\\
&+ \frac{\Gamma(\beta+1)}{2\Delta^{\beta}_{w}(1)}\left[\mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}+;w}G\left(\frac{\kappa+\sigma}{2},\varsigma\right) + \mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}-;w}G\left(\frac{\kappa+\sigma}{2},\varphi\right)\right]\\
&\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2\Lambda^{\alpha}_{g}(1)\Delta^{\beta}_{w}(1)}\Big[\mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}F(\sigma,\varsigma) + \mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-;g,w}F(\sigma,\varphi)\\
&\quad + \mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+;g,w}F(\kappa,\varsigma) + \mathcal{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-;g,w}F(\kappa,\varphi)\Big]\\
&\leq \frac{\Gamma(\alpha+1)}{4\Lambda^{\alpha}_{g}(1)}\Big[\mathcal{J}^{\alpha}_{\frac{\kappa+\sigma}{2}+;g}H(\sigma,\varphi) + \mathcal{J}^{\alpha}_{\frac{\kappa+\sigma}{2}+;g}H(\sigma,\varsigma)\\
&\quad + \mathcal{J}^{\alpha}_{\frac{\kappa+\sigma}{2}-;g}H(\kappa,\varphi) + \mathcal{J}^{\alpha}_{\frac{\kappa+\sigma}{2}-;g}H(\kappa,\varsigma)\Big]\\
&+ \frac{\Gamma(\beta+1)}{4\Delta^{\beta}_{w}(1)}\Big[\mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}+;w}G(\kappa,\varsigma) + \mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}+;w}G(\sigma,\varsigma)\\
&\quad + \mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}-;w}G(\kappa,\varphi) + \mathcal{J}^{\beta}_{\frac{\varphi+\varsigma}{2}-;w}G(\sigma,\varphi)\Big].
\end{aligned}
$$

Thus, the second and third inequalities in (2.16) are established. On the other hand, from the first inequality in (1.5), we have

$$
\begin{aligned}
\varphi\left(\frac{\kappa+\sigma}{2}\right) \leq \frac{\alpha}{2\Lambda^{\alpha}_{g}(1)}\Big[&\int_{\frac{\kappa+\sigma}{2}}^{\sigma}\frac{g'(\varpi)}{[g(\sigma)-g(\varpi)]^{1-\alpha}}\big(\varphi(\varpi)+\varphi(\kappa+\sigma-\varpi)\big)\varsigma\varpi\\
&+\int_{\kappa}^{\frac{\kappa+\sigma}{2}}\frac{g'(\varpi)}{[g(\varpi)-g(\kappa)]^{1-\alpha}}\big(\varphi(\varpi)+\varphi(\kappa+\sigma-\varpi)\big)\varsigma\varpi\Big].
\end{aligned}
\tag{2.29}
$$

Since F is co-ordinated convex on Δ, applying the inequality (2.29), we have

$$
\begin{aligned}
F\left(\frac{\kappa+\sigma}{2},\frac{\varphi+\varsigma}{2}\right) \leq \frac{\alpha}{2\Lambda^{\alpha}_{g}(1)}\Big[&\int_{\frac{\kappa+\sigma}{2}}^{\sigma}\frac{g'(\varpi)}{[g(\sigma)-g(\varpi)]^{1-\alpha}}\left(F\left(\varpi,\frac{\varphi+\varsigma}{2}\right)+F\left(\kappa+\sigma-\varpi,\frac{\varphi+\varsigma}{2}\right)\right)\varsigma\varpi\\
&+\int_{\kappa}^{\sigma}\frac{g'(\varpi)}{[g(\varpi)-g(\kappa)]^{1-\alpha}}\left(F\left(\varpi,\frac{\varphi+\varsigma}{2}\right)+F\left(\kappa+\sigma-\varpi,\frac{\varphi+\varsigma}{2}\right)\right)\varsigma\varpi\Big]\\
=\frac{\Gamma(\alpha+1)}{2\Lambda^{\alpha}_{g}(1)}&\left[\mathcal{J}^{\alpha}_{\frac{\kappa+\sigma}{2}+;g}H\left(\sigma,\frac{\varphi+\varsigma}{2}\right)+\mathcal{J}^{\alpha}_{\frac{\kappa+\sigma}{2}-;g}H\left(\kappa,\frac{\varphi+\varsigma}{2}\right)\right].
\end{aligned}
\tag{2.30}
$$

Similarly, we establish

$$
\begin{aligned}
F\left(\frac{\kappa+\sigma}{2},\frac{\varphi+\varsigma}{2}\right) \leq \frac{\beta}{2\Delta^{\beta}_{w}(1)}\Big[&\int_{\frac{\varphi+\varsigma}{2}}^{\varsigma}\frac{w'(\psi)}{[w(\varsigma)-w(\psi)]^{1-\beta}}\left(F\left(\frac{\kappa+\sigma}{2},\psi\right)+F\left(\frac{\kappa+\sigma}{2},\varphi+\varsigma-\psi\right)\right)\varsigma\psi\\
&+\int_{\varphi}^{\frac{\varphi+\varsigma}{2}}\frac{w'(\psi)}{[w(\psi)-w(\varphi)]^{1-\beta}}\left(F\left(\frac{\kappa+\sigma}{2},\psi\right)+F\left(\frac{\kappa+\sigma}{2},\varphi+\varsigma-\psi\right)\right)\varsigma\psi\Big]
\end{aligned}
$$

$$= \frac{\Gamma(\beta+1)}{2\Delta_w^\beta(1)}\left[\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}+;w}G\left(\frac{\kappa+\sigma}{2},\varsigma\right)+\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}-;w}G\left(\frac{\kappa+\sigma}{2},\varphi\right)\right]. \tag{2.31}$$

By combining inequalities (2.30) and (2.31), we establish the first inequality in (2.16). From the second inequality in (1.5), we get

$$\begin{aligned}&\frac{\alpha}{2\Lambda_g^\alpha(1)}\Bigg[\int_{\frac{\kappa+\sigma}{2}}^{\sigma}\frac{g'(\varpi)}{[g(\sigma)-g(\varpi)]^\alpha}\big(\varphi(\varpi)+\varphi(\kappa+\sigma-\varpi)\big)\varsigma\varpi\\&+\int_{\kappa}^{\frac{\kappa+\sigma}{2}}\frac{g'(\varpi)}{[g(\varpi)-g(\kappa)]^\alpha}\big(\varphi(\varpi)+\varphi(\kappa+\sigma-\varpi)\big)\varsigma\varpi\Bigg]\le\frac{\varphi(\kappa)+\varphi(\sigma)}{2}.\end{aligned} \tag{2.32}$$

Using (2.32), we establish the following inequalities:

$$\frac{\Gamma(\alpha+1)}{2\Lambda_g^\alpha(1)}\left[\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}+;g}H(\sigma,\varphi)+\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}-;g}H(\kappa,\varphi)\right]\le\frac{F(\kappa,\varphi)+F(\sigma,\varphi)}{2}, \tag{2.33}$$

$$\frac{\Gamma(\alpha+1)}{2\Lambda_g^\alpha(1)}\left[\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}+;g}H(\sigma,\varsigma)+\mathcal{J}^\alpha_{\frac{\kappa+\sigma}{2}-;g}H(\kappa,\varsigma)\right]\le\frac{F(\kappa,\varsigma)+F(\sigma,\varsigma)}{2}, \tag{2.34}$$

$$\frac{\Gamma(\beta+1)}{2\Delta_w^\beta(1)}\left[\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}+;w}G(\kappa,\varsigma)+\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}-;w}G(\kappa,\varphi)\right]\le\frac{F(\kappa,\varphi)+F(\kappa,\varsigma)}{2}, \tag{2.35}$$

$$\frac{\Gamma(\beta+1)}{2\Delta_w^\beta(1)}\left[\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}+;w}G(\sigma,\varsigma)+\mathcal{J}^\beta_{\frac{\varphi+\varsigma}{2}-;w}G(\sigma,\varphi)\right]\le\frac{F(\sigma,\varphi)+F(\sigma,\varsigma)}{2}. \tag{2.36}$$

Thus, by combining inequalities (2.33)-(2.36), we arrive at the last inequality in (2.16), thereby completing the proof. □

Corollary 2.4. *Let $g(t)=t$ and $w(s)=s$ for Theorem 2.2. Then, the following inequalities for Riemann-Liouville fractional integrals hold*

$$\begin{aligned}F\left(\frac{\kappa+\sigma}{2},\frac{\varphi+\varsigma}{2}\right)&\le\frac{2^{\alpha-2}\Gamma(\alpha+1)}{(\sigma-\kappa)^\alpha}\left[J^\alpha_{\frac{\kappa+\sigma}{2}+}F\left(\sigma,\frac{\varphi+\varsigma}{2}\right)+J^\alpha_{\frac{\kappa+\sigma}{2}-}F\left(\kappa,\frac{\varphi+\varsigma}{2}\right)\right]\\&\quad+\frac{2^{\beta-2}\Gamma(\beta+1)}{(\varsigma-\varphi)^\beta}\left[J^\beta_{\frac{\varphi+\varsigma}{2}+}F\left(\frac{\kappa+\sigma}{2},\varsigma\right)+J^\beta_{\frac{\varphi+\varsigma}{2}-}F\left(\frac{\kappa+\sigma}{2},\varphi\right)\right]\\&\le\frac{2^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\sigma-\kappa)^\alpha(\varsigma-\varphi)^\beta}\\&\quad\times\left[J^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+}F(\sigma,\varsigma)+J^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-}F(\sigma,\varphi)\right.\\&\quad\left.+J^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+}F(\kappa,\varsigma)+J^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-}F(\kappa,\varphi)\right]\\&\le\frac{2^{\alpha-3}\Gamma(\alpha+1)}{(\sigma-\kappa)^\alpha}\left[J^\alpha_{\frac{\kappa+\sigma}{2}+}F(\sigma,\varphi)+J^\alpha_{\frac{\kappa+\sigma}{2}+}F(\sigma,\varsigma)\right.\\&\quad\left.+J^\alpha_{\frac{\kappa+\sigma}{2}-}F(\kappa,\varphi)+J^\alpha_{\frac{\kappa+\sigma}{2}-}F(\kappa,\varsigma)\right]\\&\quad+\frac{2^{\beta-3}\Gamma(\beta+1)}{(\varsigma-\varphi)^\beta}\left[J^\beta_{\frac{\varphi+\varsigma}{2}+}F(\kappa,\varsigma)+J^\beta_{\frac{\varphi+\varsigma}{2}+}F(\sigma,\varsigma)\right.\\&\quad\left.+J^\beta_{\frac{\varphi+\varsigma}{2}-}F(\kappa,\varphi)+J^\beta_{\frac{\varphi+\varsigma}{2}-}F(\sigma,\varphi)\right]\\&\le\frac{F(\kappa,\varphi)+F(\kappa,\varsigma)+F(\sigma,\varphi)+F(\sigma,\varsigma)}{4}.\end{aligned} \tag{2.37}$$

Example 2.2. Consider the function $F : [2,4] \times [2,4] \to \mathbb{R}$ defined by

$$F(t,s) = t^2 s^2.$$

Using the first term in inequality (2.37), we calculate

$$F\left(\frac{\kappa+\sigma}{2}, \frac{\varphi+\varsigma}{2}\right) = F(3,3) = 81.$$

For the second term in (2.37), we obtain

$$\begin{aligned}
&\frac{2^{\alpha-2}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}\left[J^{\alpha}_{3+}F(4,3) + J^{\alpha}_{3-}F(2,3)\right] \\
&\quad + \frac{2^{\beta-2}\Gamma(\beta+1)}{(\varsigma-\varphi)^{\beta}}\left[J^{\beta}_{3+}F(3,4) + J^{\beta}_{3-}F(3,2)\right] \\
&= \frac{\Gamma(\alpha+1)}{4}\left\{\frac{1}{\Gamma(\alpha)}\left[\int_3^4 9(4-t)^{\alpha-1}t^2\varsigma t + \int_2^3 9(t-2)^{\alpha-1}t^2\varsigma t\right]\right\} \\
&\quad + \frac{\Gamma(\beta+1)}{4}\left\{\frac{1}{\Gamma(\beta)}\left[\int_3^4 9(4-s)^{\beta-1}s^2\varsigma s + \int_2^3 9(s-2)^{\beta-1}s^2\varsigma s\right]\right\} \\
&= \frac{9}{2}\left(\frac{9\alpha^2+27\alpha+20}{(\alpha+1)(\alpha+2)}\right) + \frac{9}{2}\left(\frac{9\beta^2+27\beta+20}{(\beta+1)(\beta+2)}\right).
\end{aligned}$$

For the third term in (2.37), we compute

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left[J^{\alpha,\beta}_{3+,3+}F(4,4) + J^{\alpha,\beta}_{3+,3-}F(4,2) + J^{\alpha,\beta}_{3-,3+}F(2,4) + J^{\alpha,\beta}_{3-,3-}F(2,2)\right] \\
&= \left(\frac{9\alpha^2+27\alpha+20}{(\alpha+1)(\alpha+2)}\right)\left(\frac{9\beta^2+27\beta+20}{(\beta+1)(\beta+2)}\right).
\end{aligned}$$

For the fourth term in (2.37), we evaluate

$$\begin{aligned}
&\frac{2^{\alpha-3}\Gamma(\alpha+1)}{(\sigma-\kappa)^{\alpha}}\left[J^{\alpha}_{3+}F(4,2) + J^{\alpha}_{3+}F(4,4) + J^{\alpha}_{3-}F(2,2) + J^{\alpha}_{3-}F(2,4)\right] \\
&\quad + \frac{2^{\beta-3}\Gamma(\beta+1)}{(\varsigma-\varphi)^{\beta}}\left[J^{\beta}_{3+}F(2,4) + J^{\beta}_{3+}F(4,4) + J^{\beta}_{3-}F(2,2) + J^{\beta}_{3-}F(4,2)\right] \\
&= 5\left(\frac{9\alpha^2+27\alpha+20}{(\alpha+1)(\alpha+2)}\right) + 5\left(\frac{9\beta^2+27\beta+20}{(\beta+1)(\beta+2)}\right).
\end{aligned}$$

For the final term in (2.37), we compute

$$\begin{aligned}
&\frac{F(\kappa,\varphi) + F(\kappa,\varsigma) + F(\sigma,\varphi) + F(\sigma,\varsigma)}{4} \\
&= \frac{F(2,2) + F(2,4) + F(4,2) + F(4,4)}{4} = 100.
\end{aligned}$$

As a result, we obtain

$$\begin{aligned}
81 &< \frac{9}{2}\left(\frac{9\alpha^2+27\alpha+20}{(\alpha+1)(\alpha+2)}\right) + \frac{9}{2}\left(\frac{9\beta^2+27\beta+20}{(\beta+1)(\beta+2)}\right) \\
&< \left(\frac{9\alpha^2+27\alpha+20}{(\alpha+1)(\alpha+2)}\right)\left(\frac{9\beta^2+27\beta+20}{(\beta+1)(\beta+2)}\right) \\
&< 5\left(\frac{9\alpha^2+27\alpha+20}{(\alpha+1)(\alpha+2)}\right) + 5\left(\frac{9\beta^2+27\beta+20}{(\beta+1)(\beta+2)}\right) < 100
\end{aligned} \tag{2.38}$$

Figure 2 illustrates the graphical verification of the validity of inequality 2.38, where the parameters α and β are considered within the interval $[0,6]$.

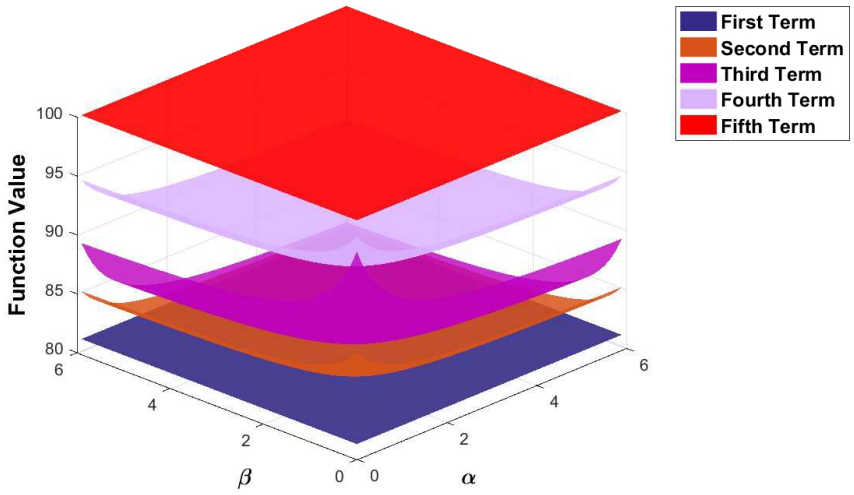

Figure 2. Graph of Example 2.2 plotted using MATLAB to illustrate its variation with respect to the parameters α and β, where $\alpha, \beta \in [0, 6]$.

Remark 2.2. By taking $\alpha = 1$ and $\beta = 1$ for Corollary 2.4, we obtain the following inequalities

$$\begin{aligned} F\left(\frac{\kappa+\sigma}{2}, \frac{\varphi+\varsigma}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{\sigma-\kappa}\int_{\kappa}^{\sigma} F\left(t, \frac{\varphi+\varsigma}{2}\right)\varsigma t + \frac{1}{\varsigma-\varphi}\int_{\varphi}^{\varsigma} F\left(\frac{\kappa+\sigma}{2}, s\right)\varsigma s\right] \\ &\leq \frac{1}{(\sigma-\kappa)(\varsigma-\varphi)}\int_{\kappa}^{\sigma}\int_{\varphi}^{\varsigma} F(t,s)\,\varsigma s\,\varsigma t \\ &\leq \frac{1}{4(\sigma-\kappa)}\left[\int_{\kappa}^{\sigma} F(\varpi,\varphi)\,\varsigma\varpi + \int_{\kappa}^{\sigma} F(\varpi,\varsigma)\,\varsigma\varpi\right] \\ &\quad + \frac{1}{4(\varsigma-\varphi)}\left[\int_{\varphi}^{\varsigma} F(\kappa,\psi)\,\varsigma\psi + \int_{\varphi}^{\varsigma} F(\sigma,\psi)\,\varsigma\psi\right] \\ &\leq \frac{F(\kappa,\varphi)+F(\kappa,\varsigma)+F(\sigma,\varphi)+F(\sigma,\varsigma)}{4}. \end{aligned}$$

This result was previously established by Dragomir in [13].

Corollary 2.5. *Under the assumptions of Theorem 2.2, if we take $g(t) = \ln t$ and $w(s) = \ln s$, then we obtain the following inequalities for Hadamard fractional integrals*

$$\begin{aligned} F\left(\frac{\kappa+\sigma}{2}, \frac{\varphi+\varsigma}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{4\Lambda_{\ln}^{\alpha}(1)}\left[\mathbf{J}^{\alpha}_{\frac{\kappa+\sigma}{2}+;g}H\left(\sigma, \frac{\varphi+\varsigma}{2}\right) + \mathbf{J}^{\alpha}_{\frac{\kappa+\sigma}{2}-;g}H\left(\kappa, \frac{\varphi+\varsigma}{2}\right)\right] \\ &\quad + \frac{\Gamma(\beta+1)}{4\Delta_{\ln}^{\beta}(1)}\left[\mathbf{J}^{\beta}_{\frac{\varphi+\varsigma}{2}+;w}G\left(\frac{\kappa+\sigma}{2}, \varsigma\right) + \mathbf{J}^{\beta}_{\frac{\varphi+\varsigma}{2}-;w}G\left(\frac{\kappa+\sigma}{2}, \varphi\right)\right] \\ &\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\Lambda_{\ln}^{\alpha}(1)\Delta_{\ln}^{\beta}(1)} \\ &\quad \times \left[\mathbf{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+;g,w}F(\sigma,\varsigma) + \mathbf{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-;g,w}F(\sigma,\varphi)\right. \\ &\quad \left. + \mathbf{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+;g,w}F(\kappa,\varsigma) + \mathbf{J}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-;g,w}F(\kappa,\varphi)\right] \\ &\leq \frac{\Gamma(\alpha+1)}{8\Lambda_{\ln}^{\alpha}(1)}\left[\mathbf{J}^{\alpha}_{\frac{\kappa+\sigma}{2}+;g}H(\sigma,\varphi) + \mathbf{J}^{\alpha}_{\frac{\kappa+\sigma}{2}+;g}H(\sigma,\varsigma)\right. \\ &\quad \left. + \mathbf{J}^{\alpha}_{\frac{\kappa+\sigma}{2}-;g}H(\kappa,\varphi) + \mathbf{J}^{\alpha}_{\frac{\kappa+\sigma}{2}-;g}H(\kappa,\varsigma)\right] \end{aligned}$$

$$
\begin{aligned}
&+\frac{\Gamma(\beta+1)}{8\Delta_{\ln}^{\beta}(1)}\left[\mathbf{J}^{\beta}_{\frac{\varphi+\varsigma}{2}+;w}G(\kappa,\varsigma)+\mathbf{J}^{\beta}_{\frac{\varphi+\varsigma}{2}+;w}G(\sigma,\varsigma)\right.\\
&\left.+\mathbf{J}^{\beta}_{\frac{\varphi+\varsigma}{2}-;w}G(\kappa,\varphi)+\mathbf{J}^{\beta}_{\frac{\varphi+\varsigma}{2}-;w}G(\sigma,\varphi)\right]\\
&\leq\frac{F(\kappa,\varphi)+F(\kappa,\varsigma)+F(\sigma,\varphi)+F(\sigma,\varsigma)}{4}.
\end{aligned}
$$

Corollary 2.6. *Under the assumptions of Theorem 2.2, if we take $g(t)=\frac{t^{\rho}}{\rho}$ and $w(s)=\frac{s^{\sigma}}{\sigma}$, then we obtain the following inequalities for Katugampola fractional integrals*

$$
\begin{aligned}
F\left(\frac{\kappa+\sigma}{2},\frac{\varphi+\varsigma}{2}\right)&\leq\frac{\Gamma(\alpha+1)}{4\Lambda_{\rho}^{\alpha}(1)}\left[{}^{\rho}\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}+}H\left(\sigma,\frac{\varphi+\varsigma}{2}\right)+{}^{\rho}\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}-}H\left(\kappa,\frac{\varphi+\varsigma}{2}\right)\right]\\
&\quad+\frac{\Gamma(\beta+1)}{4\Delta_{\sigma}^{\beta}(1)}\left[{}^{\sigma}\mathcal{I}^{\beta}_{\frac{\varphi+\varsigma}{2}+}G\left(\frac{\kappa+\sigma}{2},\varsigma\right)+{}^{\sigma}\mathcal{I}^{\beta}_{\frac{\varphi+\varsigma}{2}-}G\left(\frac{\kappa+\sigma}{2},\varphi\right)\right]\\
&\leq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\Lambda_{\rho}^{\alpha}(1)\Delta_{\sigma}^{\beta}(1)}\\
&\quad\times\left[{}^{\rho,\sigma}\mathcal{I}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}+}F(\sigma,\varsigma)+{}^{\rho,\sigma}\mathcal{I}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}+,\frac{\varphi+\varsigma}{2}-}F(\sigma,\varphi)\right.\\
&\quad\left.+{}^{\rho,\sigma}\mathcal{I}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}+}F(\kappa,\varsigma)+{}^{\rho,\sigma}\mathcal{I}^{\alpha,\beta}_{\frac{\kappa+\sigma}{2}-,\frac{\varphi+\varsigma}{2}-}F(\kappa,\varphi)\right]\\
&\leq\frac{\Gamma(\alpha+1)}{8\Lambda_{\rho}^{\alpha}(1)}\left[{}^{\rho}\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}+}H(\sigma,\varphi)+{}^{\rho}\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}+}H(\sigma,\varsigma)\right.\\
&\quad\left.+{}^{\rho}\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}-}H(\kappa,\varphi)+{}^{\rho}\mathcal{I}^{\alpha}_{\frac{\kappa+\sigma}{2}-}H(\kappa,\varsigma)\right]\\
&\quad+\frac{\Gamma(\beta+1)}{8\Delta_{\sigma}^{\beta}(1)}\left[{}^{\sigma}\mathcal{I}^{\beta}_{\frac{\varphi+\varsigma}{2}+}G(\kappa,\varsigma)+{}^{\sigma}\mathcal{I}^{\beta}_{\frac{\varphi+\varsigma}{2}+}G(\sigma,\varsigma)\right.\\
&\quad\left.+{}^{\sigma}\mathcal{I}^{\beta}_{\frac{\varphi+\varsigma}{2}-}G(\kappa,\varphi)+{}^{\sigma}\mathcal{I}^{\beta}_{\frac{\varphi+\varsigma}{2}-}G(\sigma,\varphi)\right]\\
&\leq\frac{F(\kappa,\varphi)+F(\kappa,\varsigma)+F(\sigma,\varphi)+F(\sigma,\varsigma)}{4}.
\end{aligned}
$$

3. Conclusion

In this investigation, we establish Hermite-Hadamard type inequalities, including generalized fractional integrals for co-ordinated convex functions. We obtain several novel results that generalize classical integral inequalities by utilizing the concepts of Riemann-Liouville, Hadamard, and Katugampola fractional integrals. We acquired that the obtained inequalities extend well-known results in the literature. In particular, when $\alpha=\beta=1$, our results reduce to their classical counterparts. The theoretical results presented in this paper have potential applications in various areas, such as mathematical physics, optimization theory, and numerical analysis. In future work, these inequalities could be further explored in different fractional integral settings, including generalized fractional operators and integral transforms. Additionally, investigating the impact of different convexity classes may lead to further refining the existing results.

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