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Hermite-Hadamard type inequalities for new conditions on h-convex functions via ψ-Hilfer integral operators

Bouharket Benaissa[1] · **Noureddine Azzouz**[2,4] · **Hüseyin Budak**[3]

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Abstract
We employ a new function class called B-function to create a new version of fractional Hermite–Hadamard and trapezoid type inequalities on the right-hand side that involves h-convex and ψ-Hilfer operators. We also provide new midpoint-type inequalities using h-convex functions.

Keywords Fractional conformable integrals · Fractional conformable derivative · Hermite–Hadamard inequality

Mathematics Subject Classification Primary 26D10 · 26A33 · 26D15

1 Introduction

Convexity theory provides robust approaches and concepts for solving an extensive variety of problems in pure and practical mathematics. Because of their tenacity, convex functions have been applied to a wide range of mathematical disciplines, resulting in the discovery of several inequalities in the literature. One famous inequality

✉ Hüseyin Budak
hsyn.budak@gmail.com

Bouharket Benaissa
bouharket.benaissa@univ-tiaret.dz

Noureddine Azzouz
n.azzouz@cu-elbayadh.dz

1 Laboratory of Informatics and Mathematics, Faculty of Material Sciences, University of Tiaret, Tiaret, Algeria

2 Faculty of Sciences, University Center Nour Bachir El Bayadh, El Bayadh, Algeria

3 Department of Mathematics Faculty of Science and Arts, Düzce University, 81620 Düzce, Turkey

4 University Belhadj Bouchaib Ain Temouchent, Ain Temouchent, Algeria

is the Hermite–Hadamard inequality, which has found applications in convexity theory. In [16], the author introduces a novel class of functions called as h-convex functions

Definition 1 Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, if f is non-negative and for all $x, y \in I, \alpha \in (0, 1)$ we have

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha) f(x) + h(1-\alpha) f(y). \tag{1.1}$$

If the inequality (1.1) is reversed, then f is said to be h-concave.

By putting $h(\alpha) = \alpha, h(\alpha) = 1, h(\alpha) = \alpha^s, h(\alpha) = \frac{1}{n}\sum_{k=1}^{n} \alpha^{\frac{1}{k}}$ and $h(\alpha) = \frac{1}{\alpha}$ in (1.1), Definition 1 reduces to some wel known function classes, P-functions [5, 11], s-convex functions [3], n-fractional polynomial convex functions [7] and Godunova–Levin functions [6], respectively.

2 ψ-Hilfer operators

Definition 2 Let $[a, b] \subseteq [0, +\infty)$. Let $\beta > 0$ and ψ be a positive, strictly increasing differentiable function such that $\psi'(\tau) \neq 0$ for all $\tau \in [a, b]$. The left and right sided ψ-Hilfer fractional integral of a function f with respect to the function ψ on $[a, b]$ are defined respectively as follows.

$$^{\psi}\mathfrak{J}^{\beta}_{a+} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (\psi(x) - \psi(t))^{\beta-1} \psi'(t) f(t) dt, \quad a < x \leq b. \tag{2.1}$$

$$^{\psi}\mathfrak{J}^{\beta}_{b-} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b (\psi(t) - \psi(x))^{\beta-1} \psi'(t) f(t) dt, \quad a \leq x < b, \tag{2.2}$$

where the gamma function verified

$$\Gamma(\beta) = \int_0^{\infty} t^{\beta-1} e^{-t} dt\,, \quad \beta\,\Gamma(\beta) = \Gamma(\beta+1).$$

For these operators, consider the following space

$$X[a, b] = \left\{ f : \| f \|_X = \left(\int_a^b | f(x) | \psi'(x) dx \right) < \infty \right\}.$$

One essential property of ψ-Hilfer operators is that they are dependent on the function ψ and produce a particular type of fractional integrals.

(1) Taking $\psi(\tau) = \tau$, we get Riemann-Liouville fractional operator of order $\beta > 0$

$$\mathcal{RL}^{\beta}_{a+} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-s)^{\beta-1} f(s) ds, \quad x > a,$$

$$\mathcal{RL}^{\beta}_{b-} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b (s-x)^{\beta-1} f(s) ds,, \quad x < b.$$

(2) Using $\psi(\tau) = \ln \tau$, we deduce Hadamard fractional operator of order $\beta > 0$

$$\mathcal{H}^{\beta}_{a^+} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left(\ln \frac{x}{s}\right)^{\beta-1} f(s) \frac{ds}{s}, \quad x > a > 1,$$
$$\mathcal{H}^{\beta}_{a^+} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \left(\ln \frac{s}{x}\right)^{\beta-1} f(s) \frac{ds}{s}, \quad 1 < x < b.$$

(3) Putting $\psi(\tau) = \frac{\tau^{\rho}}{\rho}$ where $\rho > 0$, we obtain Katugompola fractional operators of order $\beta > 0$.

$$\mathcal{K}^{\beta}_{a^+} f(x) = \frac{(\rho)^{1-\beta}}{\Gamma(\beta)} \int_a^x \left(x^{\rho} - s^{\rho}\right)^{\beta-1} f(s) s^{\rho-1} ds, \ x > a,$$
$$\mathcal{K}^{\beta}_{b^-} f(x) = \frac{(\rho)^{1-\beta}}{\Gamma(\beta)} \int_x^b \left(s^{\rho} - x^{\rho}\right)^{\beta-1} f(s) s^{\rho-1} ds, \ x < b.$$

In the literature, papers devoted to fractional integral inequalities. For some of them, please refer to [1, 2, 4, 9, 12–14].

3 *B*-function

We now give a definition of the B-function.

Definition 3 Let $a < b$ and $h : (a, b) \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function. The function h is a B-function, or that h belongs to the class $B(a, b)$, if for all $x \in (a, b)$, we have

$$h(x - a) + h(b - x) \leq 2\, h\left(\frac{a + b}{2}\right). \tag{3.1}$$

If the inequality (3.1) is reversed, h is called A-function, or that h belongs to the class $A(a, b)$.
If we have the equality in (3.1), h is called AB-function, or that h belongs to the class $AB(a, b)$.

Corollary 1 *Let $h : (0, 1) \to \mathbb{R}$ be a non-negative function.*

(1) *The function $h(\alpha) = 1$ is AB-function, B-function and A-function. And the function $h(\alpha) = \alpha$ is AB-function, B-function and A-function under the conditions $a = 0$, $a < 0$ and $a > 0$, respectively.*
(2) *The function $h(\alpha) = \alpha^s$, $s \in (0, 1]$ is B-function.*
(3) *The function $h(\alpha) = \frac{1}{n} \sum_{k=1}^{n} \alpha^{\frac{1}{k}}$, $n, k \in \mathbb{N}$ is B-function.*
(4) *The function $h(\alpha) = \frac{1}{\alpha}$ is A-function.*

Proof (1) The first case is obvious.
The following result (3.2) is required to prove the next case.
Let $g(x) = \ln(x + 1)$, $g(0) = 0$, $g(1) = \ln 2$ and $M(1, \ln 2)$. On the interval

$[0, 1]$, the graph of the function g appears over the line (OM). This gives us, for all $x \in (0, 1]$

$$\ln(x+1) \geq x \ln 2 \Leftrightarrow (1+x)^{\frac{1}{x}} \geq 2,$$

taking $x = \frac{1}{p}$, we have

$$\text{for all } p \geq 1, \quad \left(1+\frac{1}{p}\right)^{p} \geq 2\,. \tag{3.2}$$

(2) Let $h(\alpha) = \alpha^{s}$, $s \in (0, 1]$ and taking $s = \frac{1}{p}$. The function h is a B-function, this means

$$\alpha^{\frac{1}{p}} + (1-\alpha)^{\frac{1}{p}} \leq \left(\frac{1}{2}\right)^{\frac{1}{p}-1}. \tag{3.3}$$

We use absurdity to demonstrate inequality (3.3), suppose that exist $p \geq 1$ verified

$$\alpha^{\frac{1}{p}} + (1-\alpha)^{\frac{1}{p}} > \left(\frac{1}{2}\right)^{\frac{1}{p}-1},$$

we get

$$\int_0^1 \left[\alpha^{\frac{1}{p}} + (1-\alpha)^{\frac{1}{p}}\right] d\alpha > \left(\frac{1}{2}\right)^{\frac{1}{p}-1},$$

thus

$$2\left(\frac{p}{p+1}\right) > \left(\frac{1}{2}\right)^{\frac{1}{p}-1}.$$

This gives

$$\left(\frac{p}{p+1}\right)^{p} > \frac{1}{2},$$

hence

$$\left(1+\frac{1}{p}\right)^{p} < 2 \text{ which is absurd.}$$

(3) Let $h(\alpha) = \dfrac{1}{n}\sum_{k=1}^{n} \alpha^{\frac{1}{k}}$, $n, k \in \mathbb{N}$, we have

$$\alpha^{\frac{1}{k}} + (1-\alpha)^{\frac{1}{k}} \leq \left(\frac{1}{2}\right)^{\frac{1}{k}-1},$$

then

$$\frac{1}{n}\sum_{k=1}^{n} \alpha^{\frac{1}{k}} + \frac{1}{n}\sum_{k=1}^{n} (1-\alpha)^{\frac{1}{k}} \leq \frac{2}{n}\sum_{k=1}^{n} \left(\frac{1}{2}\right)^{\frac{1}{k}}.$$

This yields

$$h(\alpha) + h(1-\alpha) \leq 2\,h\left(\frac{1}{2}\right).$$

(4) Let $h(\alpha) = \frac{1}{\alpha}$ with $\alpha \in (0, 1)$, we have

$$\frac{(2\alpha - 1)^2}{\alpha(1-\alpha)} \geq 0 \Leftrightarrow \frac{1}{\alpha} + \frac{1}{1-\alpha} \geq 4,$$

therefore

$$h(\alpha) + h(1-\alpha) \geq 2\,h\left(\frac{1}{2}\right).$$

□

Motivated by previous literature, we use ψ-Hilfer operators to establish a new version of Hermite–Hadamard and trapezoid inequalities based on h-convex functions where h belongs to the class $B(0, 1)$. We also provide new midpoint inequalities using h-convex functions.

4 Hermite–Hadamard inequalities

We now present the first result of the Hermite–Hadamard inequality for h-convexity of the function f involving ψ-Hilfer operators.

Theorem 4.1 *Let $\beta > 0$, h be a B-function. Let $f \in X[a, b]$ be a h-convex function and ψ be a positive, strictly increasing differentiable function such that $\psi'(\tau) \neq 0$ for all $\tau \in [a, b]$. Then the following inequalities hold*

$$\begin{aligned}\frac{1}{2\,h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\beta+1)}{4\,(\psi(b)-\psi(a))^{\beta}}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a) + {}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right] \\ &\leq 2\,h\left(\frac{1}{2}\right)\left[\frac{f(b)+f(a)}{2}\right], \end{aligned} \tag{4.1}$$

where

$$F(t) = f(t) + f(a+b-t). \tag{4.2}$$

Proof For any $t \in [a, b]$, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}(a+b-t) + \frac{1}{2}t\right) \\ &\leq h\left(\frac{1}{2}\right) f(a+b-t) + h\left(\frac{1}{2}\right) f(t), \end{aligned}$$

then

$$f\left(\frac{a+b}{2}\right) \leq h\left(\frac{1}{2}\right) F(t). \tag{4.3}$$

Multiplying (4.3) by $\beta\,(\psi(b) - \psi(t))^{\beta-1}\,\psi'(t)$ and integrating over $t \in [a, b]$, we result

$$f\left(\frac{a+b}{2}\right)(\psi(b)-\psi(a))^{\beta} \leq \Gamma(\beta+1)h\left(\frac{1}{2}\right) {}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b). \tag{4.4}$$

Multiplying (4.3) by $\beta\left(\psi(t)-\psi(a)\right)^{\beta-1}\psi'(t)$ and integrating over $t\in[a,b]$, we get

$$f\left(\frac{a+b}{2}\right)\left(\psi(b)-\psi(a)\right)^{\beta}\leq\Gamma(\beta+1)h\left(\frac{1}{2}\right)\,{}^{\psi}\mathfrak{J}_{b^-}^{\beta}F(a). \tag{4.5}$$

By adding the inequalities (4.4) and (4.5), we deduce

$$\frac{1}{h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\beta+1)}{2\left(\psi(b)-\psi(a)\right)^{\beta}}\left[{}^{\psi}\mathfrak{J}_{b^-}^{\beta}F(a)+{}^{\psi}\mathfrak{J}_{a^+}^{\beta}F(b)\right]. \tag{4.6}$$

Put $t=(1-s)\,a+s\,b$ in (4.2) for $s\in(0,1)$ and using the h-convexity of f, we get

$$\begin{aligned}F(t)&=f\left((1-s)\,b+s\,a\right)+f\left((1-s)\,a+s\,b\right)\\&\leq h(1-s)\left[f(b)+f(a)\right]+h(s)\left[f(b)+f(a)\right]\\&=\left(h(s)+h(1-s)\right)\left[f(b)+f(a)\right],\end{aligned}$$

given that h is a B-function, we conclude

$$F(t)\leq 2\,h\left(\frac{1}{2}\right)\left[f(b)+f(a)\right]. \tag{4.7}$$

Using the same method as previously on (4.7), we obtain

$$\frac{\Gamma(\beta+1)}{2\left(\psi(b)-\psi(a)\right)^{\beta}}\left[{}^{\psi}\mathfrak{J}_{b^-}^{\beta}F(a)+{}^{\psi}\mathfrak{J}_{a^+}^{\beta}F(b)\right]\leq 2\,h\left(\frac{1}{2}\right)\left[f(b)+f(a)\right]. \tag{4.8}$$

Combining the inequality (4.6) with the inequality (4.8) yields the desired outcome. □

Taking $\psi(x)=x$ and $\beta=1$, we conclude the following new version of the Hermite–Hadamard inequality, see (Theorem.1 [15]).

Corollary 2 *Let h be a B-function. Let $f\in L[a,b]$ be a h-convex function. Then the following inequalities hold*

$$\frac{1}{2\,h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)dt\leq\left[f(b)+f(a)\right]h\left(\frac{1}{2}\right). \tag{4.9}$$

The following results are dependent on the function h presented in Theorem 4.1. First, assuming $h(\alpha)=\alpha$, we may obtain the result using the convex functions described in [8].

Corollary 3 *Let $f\in X[a,b]$ be a convex function and ψ be a positive, strictly increasing differentiable function such that $\psi'(\tau)\neq 0$ for all $\tau\in[a,b]$, $\beta>0$. Then the following inequalities hold*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\beta+1)}{4\left(\psi(b)-\psi(a)\right)^{\beta}}\left[{}^{\psi}\mathfrak{J}_{b^-}^{\beta}F(a)+{}^{\psi}\mathfrak{J}_{a^+}^{\beta}F(b)\right]\leq\left[\frac{f(b)+f(a)}{2}\right], \tag{4.10}$$

where $F(t) = f(t) + f(a+b-t)$.

By setting $h(\alpha) = 1$, we get the following result using ψ-Hilfer operators with function f belongs to class P-function.

Corollary 4 *Let* $\beta > 0$, $f \in X[a,b]$ *be a* P*-function and* ψ *be a positive, strictly increasing differentiable function such that* $\psi'(\tau) \neq 0$ *for all* $\tau \in [a,b]$. *Then the following inequalities hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\beta+1)}{2\,(\psi(b)-\psi(a))^{\beta}}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a) + {}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right] \leq 2\,[f(b)+f(a)]\,, \tag{4.11}$$

where $F(t) = f(t) + f(a+b-t)$.

Using $h(\alpha) = \alpha^s$, we obtain the following result through ψ-Hilfer operators and s-convex functions.

Corollary 5 *Let* $\beta > 0$, $s \in (0,1]$ *and* $f \in X[a,b]$ *be a* s*-convex function and let* ψ *be a positive, strictly increasing differentiable function such that* $\psi'(\tau) \neq 0$ *for all* $\tau \in [a,b]$. *Then the following inequalities hold*

$$\begin{aligned} 2^{s-1} f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\beta+1)}{4\,(\psi(b)-\psi(a))^{\beta}}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a) + {}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right] \\ &\leq \left[\frac{f(b)+f(a)}{2^s}\right], \end{aligned} \tag{4.12}$$

where $F(t) = f(t) + f(a+b-t)$.

Taking $h(\alpha) = \frac{1}{n}\sum_{k=1}^{n}\alpha^{\frac{1}{k}}$, we deduce the following result through ψ-Hilfer operators and n-fractional polynomial convex functions.

Corollary 6 *Let* $\beta > 0$ *and* $f \in X[a,b]$ *be a* n*-fractional polynomial convex function and* ψ *be a positive, strictly increasing differentiable function such that* $\psi'(\tau) \neq 0$ *for all* $\tau \in [a,b]$. *Then the following inequalities hold*

$$\begin{aligned} \frac{1}{C_n} f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\beta+1)}{2\,(\psi(b)-\psi(a))^{\beta}}\left[{}^{\psi}\mathfrak{J}^{\alpha}_{b^-}F(a) + {}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right] \\ &\leq C_n\left[\frac{f(b)+f(a)}{2}\right], \end{aligned} \tag{4.13}$$

where $F(t) = f(t) + f(a+b-t)$ *and* $C_n = \frac{2}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}$.

Remark 1 If we choose $\psi(\tau) = \tau$, $\psi(\tau) = \ln\tau$ and $\psi(\tau) = \frac{\tau^{\rho}}{\rho}$, respectively, in Corollaries 4, 5 and 6, we obtain Hermite–Hadamard inequality for P-function, s-convex functions and n-fractional polynomial convex functions, respectively, involving Riemann-Liouville fractional operator, Hadamard fractional operator, Katugompola fractional operators, respectively.

5 Trapezoid type inequalities

In this section, we will explore certain trapezoid-like inequalities for h-convexity functions utilizing ψ-Hilfer operators, as well as their particular results, where h belongs to the class $B(0,1)$. To accomplish this, we must first establish equality in the lemma that follows.

Lemma 5.1 *If β, ψ are defined as in Theorem 4.1 and $f:[a,b]\to\mathbb{R}$ is a differentiable mapping to (a,b), then the following identity holds.*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{4\ (\psi(b)-\psi(a))^{\beta}}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]=\frac{b-a}{4\ (\psi(b)-\psi(a))^{\beta}}$$
$$\times\int_0^1\left(2\ (\psi(b)-\psi(a))^{\beta}-A_{\psi,\beta}(s)\right)\left[f'\left(sa+(1-s)b\right)-f'\left((1-s)a+sb\right)\right]ds, \tag{5.1}$$

where

$$A_{\psi,\beta}(s)=(\psi(b)-\psi\left(sa+(1-s)b\right))^{\beta}+(\psi\left((1-s)a+sb\right)-\psi(a))^{\beta}. \tag{5.2}$$

Proof Let

$$J_1=\int_a^b\left[(\psi(b)-\psi(a))^{\beta}-(\psi(b)-\psi(t))^{\beta}\right]F'(t)dt. \tag{5.3}$$

Integrating by parts (5.3) and using (4.2), we get

$$J_1=\left[(\psi(b)-\psi(a))^{\beta}-(\psi(b)-\psi(t))^{\beta}\right]F(t)\Big|_a^b-\beta\int_a^b(\psi(b)-\psi(t))^{\beta-1}\psi'(t)F(t)dt,$$

therefore

$$J_1=(\psi(b)-\psi(a))^{\beta}F(b)-\Gamma(\beta+1)\ {}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b). \tag{5.4}$$

Similarly, let

$$J_2=\int_a^b\left[(\psi(b)-\psi(a))^{\beta}-(\psi(t)-\psi(a))^{\beta}\right]F'(t)dt. \tag{5.5}$$

Integrating by parts (5.5), we deduce

$$J_2=-(\psi(b)-\psi(a))^{\beta}F(a)+\Gamma(\beta+1)\ {}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a). \tag{5.6}$$

Since $F(a)=F(b)=f(a)+f(b)$, from (5.4) and (5.6), we obtain

$$J_1-J_2=2\ (\psi(b)-\psi(a))^{\beta}(f(a)+f(b))-\Gamma(\beta+1)\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right],$$

thus

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{4\,(\psi(b)-\psi(a))^{\beta}}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]$$
$$=\frac{1}{4\,(\psi(b)-\psi(a))^{\beta}}\,(J_1-J_2)\,. \tag{5.7}$$

On the other hand, given that $F'(t)=f'(t)-f'(a+b-t)$, then from (5.3), we get

$$J_1=\int_a^b\left[(\psi(b)-\psi(a))^{\beta}-(\psi(b)-\psi(t))^{\beta}\right]\left(f'(t)-f'(a+b-t)\right)dt.$$

Changing the variable $t=sa+(1-s)b$ for $s\in[0,1]$ produces

$$J_1=(b-a)\int_0^1\left[(\psi(b)-\psi(a))^{\beta}-(\psi(b)-\psi(sa+(1-s)b))^{\beta}\right]$$
$$\times\left[f'\left(sa+(1-s)b\right)-f'\left((1-s)a+sb\right)\right]ds.$$

Similarly, from (5.5), we get

$$J_2=\int_a^b\left[(\psi(b)-\psi(a))^{\beta}-(\psi(t)-\psi(a))^{\beta}\right]\left(f'(t)-f'(a+b-t)\right)dt$$
$$=(b-a)\int_0^1\left[(\psi(b)-\psi(a))^{\beta}-(\psi((1-s)a+sb)-\psi(a))^{\beta}\right]$$
$$\times\left[f'\left((1-s)a+sb\right)-f'\left(sa+(1-s)b\right)\right]ds.$$

As a result,

$$J_1-J_2=(b-a)\int_0^1\left(2\,(\psi(b)-\psi(a))^{\beta}-A_{\psi,\beta}(s)\right)$$
$$\left[f'\left(sa+(1-s)b\right)-f'\left((1-s)a+sb\right)\right]ds. \tag{5.8}$$

To get the desired equality (5.1), simply replace (5.8) with (5.7). □

Theorem 5.1 *Assume h is a B-function and β, ψ are defined according to Lemma 5.1. If $|f'|$ is a h-convex mapping on $[a,b]$, then the trapezoid type inequality is obtained as*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]\right|$$
$$\leq\frac{(b-a)\,h\left(\frac{1}{2}\right)}{2\,\Omega(\psi,\beta)}\left[|f'(a)|+|f'(b)|\right]\int_0^1\left|2\,\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|ds, \tag{5.9}$$

where $\Omega(\psi,\beta)=(\psi(b)-\psi(a))^{\beta}$.

Proof Using the absolute value of identity (5.1), the h-convexity of the function $|f'|$ and h is a B-function, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]\right| \\
&\leq \frac{b-a}{4\,\Omega(\psi,\beta)}\int_0^1 \left|2\,\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|\left|f'\left(sa+(1-s)b\right)-f'\left((1-s)a+sb\right)\right|ds \\
&\leq \frac{b-a}{4\,\Omega(\psi,\beta)}\int_0^1 \left|2\,\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|\left(h(s)+h(1-s)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]ds \\
&\leq \frac{b-a}{4\,\Omega(\psi,\beta)}\left(2\,h\left(\frac{1}{2}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\right)\int_0^1 \left|2\,\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|ds.
\end{aligned}$$

This completes the proof. □

Using $\psi(x)=x$ and $\beta=1$, we derive the following new version of the trapezoidal inequality, see ([10, Corollary.1]).

Corollary 7 *Let h be a B-function. If $|f'|$ is a h-convex mapping on $[a,b]$, then the following inequalities hold*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{(b-a)\,h\left(\frac{1}{2}\right)}{4}\Big[|f'(a)|+|f'(b)|\Big]. \tag{5.10}$$

The following Corollaries depend on the function h given in Theorem 5.1. First, Let's put $h(\alpha)=\alpha$ to get a new version using the convex functions proven in [8].

Corollary 8 *Assume that β, ψ are defined as in Lemma 5.1. If $|f'|$ is a convex mapping on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]\right| \\
&\quad\leq\frac{b-a}{4\,\Omega(\psi,\beta)}\Big[|f'(a)|+|f'(b)|\Big]\int_0^1\left|2\,\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|ds,
\end{aligned}\tag{5.11}$$

where $\Omega(\psi,\beta)=(\psi(b)-\psi(a))^{\beta}$.

Put $h(\alpha)=1$, we get the following result via ψ-Hilfer operators with function f belongs to the class P-function.

Corollary 9 *Assume that β, ψ are defined as in Lemma 5.1. If $|f'|$ is a P-function on $[a,b]$, thus*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]\right| \\
&\quad\leq\frac{b-a}{2\,\Omega(\psi,\beta)}\Big[|f'(a)|+|f'(b)|\Big]\int_0^1\left|2\,\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|ds,
\end{aligned}\tag{5.12}$$

where $\Omega(\psi,\beta)=(\psi(b)-\psi(a))^{\beta}$.

Put $h(\alpha)=\alpha^{s}$, we get the following result via ψ-Hilfer operators with s-convex functions.

Corollary 10 *Assume that* β,ψ *are defined as in Lemma 5.1. If* $|f'|$ *is a* s*-convex mapping on* $[a,b]$*, then the trapezoid type inequality is obtained as*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]\right| \leq \frac{b-a}{2^{s+1}\,\Omega(\psi,\beta)}\Big[|f'(a)|+|f'(b)|\Big]\int_0^1\left|2\,\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|ds, \tag{5.13}$$

where $\Omega(\psi,\beta)=(\psi(b)-\psi(a))^{\beta}$.

Put $h(\alpha)=\frac{1}{n}\sum_{k=1}^{n}\alpha^{\frac{1}{k}}$, we get the following result via ψ-Hilfer operators with n-fractional polynomial convex functions.

Corollary 11 *Assume that* β,ψ *are defined as in Lemma 5.1. If* $|f'|$ *is a* n*-fractional polynomial convex mapping on* $[a,b]$*, yields*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]\right| \leq \frac{(b-a)\,C_n}{4\,\Omega(\psi,\beta)}\Big[|f'(a)|+|f'(b)|\Big]\int_0^1\left|2\,\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|ds, \tag{5.14}$$

where $\Omega(\psi,\beta)=(\psi(b)-\psi(a))^{\beta}$ *and* $C_n=\frac{2}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}$.

Remark 2 If we choose $\psi(\tau)=\tau$, $\psi(\tau)=\ln\tau$ and $\psi(\tau)=\frac{\tau^{\rho}}{\rho}$, respectively, in the corollaries 8, 9, 10 and 11, we establish trapezoid-type inequalities for the convex function, the function P, s-convex functions, and polynomial n-fractional convex functions, respectively, involving the Riemann-Liouville fractional operator, the Hadamard fractional operator, and the Katugompola fractional operators, respectively.

Theorem 5.2 *Let* h *be a B-function,* $p>1$ *and* $\frac{1}{p'}+\frac{1}{p}=1$. *Assume that* β,ψ *are defined as in Lemma 5.1. If* $\left|f'\right|^{p}$ *is a* h*-convex mapping on* $[a,b]$*, then the following trapezoid type inequality is obtained*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]\right|\\ &\leq\frac{(b-a)\left(2\,h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}}{4\,\Omega(\psi,\beta)}\left(2\int_0^1\left|2\,\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right|^{p}+\left|f'(b)\right|^{p}\right)^{\frac{1}{p}}\\ &\leq\frac{(b-a)\left(2\,h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}}{4\,\Omega(\psi,\beta)}\left(2\int_0^1\left|2\,\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right|+\left|f'(b)\right|\right),\end{aligned} \tag{5.15}$$

where $\Omega(\psi,\beta)=(\psi(b)-\psi(a))^{\beta}$.

Proof By using the absolute value on equality in Lemma 5.1, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]\right|\\
&\quad\leq\frac{b-a}{4\,\Omega(\psi,\beta)}\int_0^1\left|2\,\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|\left|f'\left(sa+(1-s)b\right)\right|ds\\
&\qquad+\frac{b-a}{4\,\Omega(\psi,\beta)}\int_0^1\left|2\,\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|\left|f'\left((1-s)a+sb\right)\right|ds,
\end{aligned}$$

Utilizing Hölder's inequality, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]\right|\\
&\leq\frac{b-a}{4\,\Omega(\psi,\beta)}\left(\int_0^1\left|2\,\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\int_0^1\left|f'\left(sa+(1-s)b\right)\right|^{p}ds\right)^{\frac{1}{p}}\\
&\quad+\frac{b-a}{4\,\Omega(\psi,\beta)}\left(\int_0^1\left|2\,\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\int_0^1\left|f'\left((1-s)a+sb\right)\right|^{p}ds\right)^{\frac{1}{p}}.
\end{aligned}$$

Notice that for $p>1$ and $A,B\geq 0$: $A^{\frac{1}{p}}+B^{\frac{1}{p}}\leq 2^{1-\frac{1}{p}}(A+B)^{\frac{1}{p}}$, then

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]\right|\\
&\quad\leq\frac{b-a}{4\,\Omega(\psi,\beta)}\left(2\int_0^1\left|2\,\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\\
&\qquad\times\left[\int_0^1\left|f'\left(sa+(1-s)b\right)\right|^{p}ds+\int_0^1\left|f'\left((1-s)a+sb\right)\right|^{p}ds\right]^{\frac{1}{p}}.
\end{aligned}$$

Given that $\left|f'\right|^p$ is h-convex function and h is B-function, we get

$$\begin{aligned}
&\int_0^1\left|f'\left(sa+(1-s)b\right)\right|^{p}ds+\int_0^1\left|f'\left((1-s)a+sb\right)\right|^{p}ds\\
&\quad\leq\left(\left|f'(a)\right|^{p}+\left|f'(b)\right|^{p}\right)\int_0^1\left(h(s)+h(1-s)\right)ds\\
&\quad\leq 2\,h\left(\frac{1}{2}\right)\left(\left|f'(a)\right|^{p}+\left|f'(b)\right|^{p}\right),
\end{aligned}$$

therefore

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]\right|$$

$$\leq \frac{(b-a)\left(2\,h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}}{4\,\Omega(\psi,\beta)}\left(2\int_0^1 \left|2\,\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right|^p+\left|f'(b)\right|^p\right)^{\frac{1}{p}}.$$

This accomplishes the first inequality in (6.14). Notice that $A^p + B^p \leq (A + B)^p$ for $p > 1$, it results the second inequality in (6.14). □

6 Midpoint type inequalities

This section presents new results on midpoint inequality involving h-convex functions via ψ-Hilfer operators, based on the identity presented in the following lemma.

Lemma 6.1 *Let $\beta > 0$, $f \in X[a,b]$ be a h-convex function and ψ be a positive, strictly increasing differentiable function such that $\psi'(\tau) \neq 0$ for all $\tau \in [a,b]$. The following identity holds*

$$\begin{aligned}\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]-f\left(\frac{a+b}{2}\right)\\ =\frac{b-a}{4\,\Omega(\psi,\beta)}\int_0^1 A_{\psi,\beta}(s)\left[f'\Big(sa+(1-s)b\Big)-f'\Big((1-s)a+sb\Big)\right]ds\\ -\frac{b-a}{2}\int_{\frac{1}{2}}^1\left[f'\Big(sa+(1-s)b\Big)-f'\Big((1-s)a+sb\Big)\right]ds. \qquad (6.1)\end{aligned}$$

Proof Let

$$K_1=-\int_a^b(\psi(b)-\psi(t))^{\beta}\,F'(t)dt+(\psi(b)-\psi(a))^{\beta}\int_a^{\frac{a+b}{2}}F'(t)dt. \qquad (6.2)$$

Integrating by parts (6.2), we get

$$\begin{aligned}K_1=&-(\psi(b)-\psi(t))^{\beta}\,F(t)\Big|_a^b-\beta\int_a^b(\psi(b)-\psi(t))^{\beta-1}\,F(t)dt\\ &+(\psi(b)-\psi(a))^{\beta}\,F(t)\Big|_a^{\frac{a+b}{2}}.\end{aligned}$$

This gives

$$K_1=(\psi(b)-\psi(a))^{\beta}\,F\left(\frac{a+b}{2}\right)-\Gamma(\beta+1)\;{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b). \qquad (6.3)$$

Consider the following integral

$$K_2=\int_a^b(\psi(t)-\psi(a))^{\beta}\,F'(t)dt-(\psi(b)-\psi(a))^{\beta}\int_{\frac{a+b}{2}}^b F'(t)dt. \qquad (6.4)$$

Integrating by parts yields

$$K_2 = (\psi(t) - \psi(a))^{\beta} F(t)\Big|_a^b - \beta \int_a^b (\psi(t) - \psi(a))^{\beta-1} F(t)dt - (\psi(b) - \psi(a))^{\beta} F(t)\Big|_{\frac{a+b}{2}}^b,$$

which gives

$$K_2 = (\psi(b) - \psi(a))^{\beta} F\left(\frac{a+b}{2}\right) - \Gamma(\beta+1)\grave{u}\, {}^{\psi}\mathfrak{J}_{b^-}^{\beta} F(a). \tag{6.5}$$

By adding (6.3) and (6.5), then using (4.2) we obtain

$$K_1 + K_2 = 4\,\Omega(\psi,\beta)\, f\left(\frac{a+b}{2}\right) - \Gamma(\beta+1)\left[{}^{\psi}\mathfrak{J}_{b^-}^{\beta} F(a) + {}^{\psi}\mathfrak{J}_{a^+}^{\beta} F(b)\right]. \tag{6.6}$$

Furthermore, by changing the variable $t = sa + (1-s)b$ in (6.2) and using (4.2), we get

$$K_1 = -(b-a)\int_0^1 (\psi(b) - \psi(sa+(1-s)b))^{\beta} \left[f'(sa+(1-s)b) - f'((1-s)a+sb)\right] ds + (b-a)\,(\psi(b)-\psi(a))^{\beta} \int_{\frac{1}{2}}^1 \left[f'(sa+(1-s)b) - f'((1-s)a+sb)\right] ds.$$

Changing the variable $t = (1-s)a + sb$ in (6.4) and using (4.2), we obtain

$$K_2 = (b-a)\int_0^1 (\psi((1-s)a+sb) - \psi(a))^{\beta} \left[f'((1-s)a+sb) - f'(sa+(1-s)b)\right] ds - (b-a)\,(\psi(b)-\psi(a))^{\beta} \int_{\frac{1}{2}}^1 \left[f'((1-s)a+sb) - f'(sa+(1-s)b)\right] ds,$$

therefore

$$K_1 + K_2 = -(b-a)\int_0^1 A_{\psi,\beta}(s)\left[f'(sa+(1-s)b) - f'((1-s)a+sb)\right] ds + 2\,(b-a)\,\Omega(\psi,\beta)\int_{\frac{1}{2}}^1 \left[f'(sa+(1-s)b) - f'((1-s)a+sb)\right] ds. \tag{6.7}$$

Replacing (6.7) in (6.6), we get the desired equality (6.1). □

Theorem 6.1 *Assume h is a B-function and β, ψ are defined according to Lemma 5.1. If $|f'|$ is a h-convex mapping on $[a, b]$, then the midpoint type inequality is achieve.*

$$\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]-f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left(1+\frac{1}{\Omega(\psi,\beta)}\int_0^1\left|A_{\psi,\beta}(s)\right|ds\right)\left[|f'(a)|+|f'(b)|\right]. \tag{6.8}$$

Proof Using the absolute value of the identity (6.1) and the h-convexity of the function $|f'|$, we get

$$\begin{aligned}
&\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4\,\Omega(\psi,\beta)}\int_0^1\left|A_{\psi,\beta}(s)\right|\left[\left|f'\big(sa+(1-s)b\big)\right|+\left|f'\big((1-s)a+sb\big)\right|\right]ds \\
&\quad+\frac{b-a}{2}\int_{\frac{1}{2}}^1\left[\left|f'\big(sa+(1-s)b\big)\right|+\left|f'\big((1-s)a+sb\big)\right|\right]ds \\
&\leq \frac{b-a}{4\,\Omega(\psi,\beta)}\int_0^1\left|A_{\psi,\beta}(s)\right|\left(h(s)+h(1-s)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]ds \\
&\quad+\frac{b-a}{2}\int_{\frac{1}{2}}^1\left(h(s)+h(1-s)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]ds.
\end{aligned}$$

Given that h is a B-function, we get

$$\begin{aligned}
&\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4\,\Omega(\psi,\beta)}\int_0^1\left|A_{\psi,\beta}(s)\right|\left(2\,h\left(\frac{1}{2}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\right)ds \\
&\quad+\frac{b-a}{2}\int_{\frac{1}{2}}^1\left(2\,h\left(\frac{1}{2}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\right)ds \\
&= \frac{(b-a)\,h\left(\frac{1}{2}\right)}{2}\left(1+\frac{1}{\Omega(\psi,\beta)}\int_0^1\left|A_{\psi,\beta}(s)\right|ds\right)\left[|f'(a)|+|f'(b)|\right].
\end{aligned}$$

This end the proof. □

Using $\psi(x)=x$ and $\beta=1$, we get the following midpoint inequality.

Corollary 12 *Assume h is a B-function. If $|f'|$ is a h-convex mapping on $[a, b]$, then the midpoint type inequality holds.*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)\right| \leq (b-a)\,h\left(\frac{1}{2}\right)\left[|f'(a)|+|f'(b)|\right]. \tag{6.9}$$

The following Corollaries depend on the function h given in Theorem 6.1. First, Let's put $h(\alpha) = \alpha$ to get midpoint inequality with convex functions.

Corollary 13 *Assume that β, ψ are defined as in Lemma 5.1. If $|f'|$ is a convex mapping on $[a, b]$, then*

$$\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a) + {}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right] - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left(1 + \frac{1}{\Omega(\psi,\beta)}\int_0^1 \left|A_{\psi,\beta}(s)\right|ds\right)\Big[|f'(a)| + |f'(b)|\Big], \tag{6.10}$$

where $\Omega(\psi, \beta) = (\psi(b) - \psi(a))^{\beta}$.

Put $h(\alpha) = 1$, we get midpoint type inequality via ψ-Hilfer operators with function f belongs to the class P-function.

Corollary 14 *Assume that β, ψ are defined as in Lemma 5.1. If $|f'|$ is a P-function on $[a, b]$, thus*

$$\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a) + {}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right] - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{2}\left(1 + \frac{1}{(\psi(b)-\psi(a))^{\beta}}\int_0^1 \left|A_{\psi,\beta}(s)\right|ds\right)\Big[|f'(a)| + |f'(b)|\Big], \tag{6.11}$$

where $\Omega(\psi, \beta) = (\psi(b) - \psi(a))^{\beta}$.

Put $h(\alpha) = \alpha^s$, we get the following midpoint type inequality involving ψ-Hilfer operators with s-convex functions.

Corollary 15 *Assume that β, ψ are defined as in Lemma 5.1. If $|f'|$ is a s-convex mapping on $[a, b]$, then the trapezoid type inequality is obtained as*

$$\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a) + {}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right] - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{2^{s+1}}\left(1 + \frac{1}{(\psi(b)-\psi(a))^{\beta}}\int_0^1 \left|A_{\psi,\beta}(s)\right|ds\right)\Big[|f'(a)| + |f'(b)|\Big], \tag{6.12}$$

where $\Omega(\psi, \beta) = (\psi(b) - \psi(a))^{\beta}$.

Put $h(\alpha) = \frac{1}{n}\sum_{k=1}^{n}\alpha^{\frac{1}{k}}$, we get midpoint type inequality via ψ-Hilfer operators with n-fractional polynomial convex functions.

Corollary 16 *Assume that β, ψ are defined as in Lemma 5.1. If $|f'|$ is a h-fractional polynomial convex mapping on $[a, b]$, yields*

$$\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a) + {}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right] - f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)\,C_n}{4}\left(1+\frac{1}{(\psi(b)-\psi(a))^{\beta}}\int_0^1 \left|A_{\psi,\beta}(s)\right|ds\right)\left[|f'(a)|+|f'(b)|\right], \tag{6.13}$$

where $\Omega(\psi,\beta)=(\psi(b)-\psi(a))^{\beta}$ *and* $C_n=\frac{2}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{k}}$.

Remark 3 If we choose $\psi(\tau)=\tau$, $\psi(\tau)=\ln\tau$ and $\psi(\tau)=\frac{\tau^{\rho}}{\rho}$, respectively, in the corollaries 13, 14, 15 and 16, we establish midpoint type inequalities for the convex function, the function P, s-convex functions, and polynomial n-fractional convex functions, respectively, involving the Riemann-Liouville fractional operator, the Hadamard fractional operator, and the Katugompola fractional operators, respectively.

Theorem 6.2 *Let* $p>1$, $\frac{1}{p'}+\frac{1}{p}=1$ *and* h *be a B-function. Assume that* β,ψ *are defined as in Lemma 6.1. If* $\left|f'\right|^{p}$ *is a h-convex mapping on* $[a,b]$*, we get the following midpoint type inequality*

$$\begin{aligned}
&\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)\left(2h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}}{2}\left(1+\frac{1}{\Omega^{p'}(\psi,\beta)}\int_0^1\left|A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left[|f'(a)|^{p}+|f'(b)|^{p}\right]^{\frac{1}{p}} \\
&\leq \frac{(b-a)\left(2h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}}{2}\left(1+\frac{1}{\Omega^{p'}(\psi,\beta)}\int_0^1\left|A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right|+\left|f'(b)\right|\right).
\end{aligned} \tag{6.14}$$

Proof Apply Lemma 5.1 and the absolute value, we get

$$\begin{aligned}
&\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4\,\Omega(\psi,\beta)}\int_0^1\left|A_{\psi,\beta}(s)\right|\left[\left|f'\big(sa+(1-s)b\big)\right|+\left|f'\big((1-s)a+sb\big)\right|\right]ds \\
&\quad+\frac{b-a}{2}\int_{\frac{1}{2}}^1\left[\left|f'\big(sa+(1-s)b\big)\right|+\left|f'\big((1-s)a+sb\big)\right|\right]ds.
\end{aligned}$$

Using Hölder inequality and $A^{\frac{1}{p}}+B^{\frac{1}{p}}\leq 2^{1-\frac{1}{p}}(A+B)^{\frac{1}{p}}$, we get

$$\begin{aligned}
&\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4\,\Omega(\psi,\beta)}\left(\int_0^1\left|A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}
\end{aligned}$$

$$\left[\left(\int_0^1 \left|f'\left(sa+(1-s)b\right)\right|^p ds\right)^{\frac{1}{p}}+\left(\int_0^1 \left|f'\left((1-s)a+sb\right)\right|^p ds\right)^{\frac{1}{p}}\right]$$

$$+\frac{b-a}{2}\left(\int_{\frac{1}{2}}^1 ds\right)^{\frac{1}{p'}}$$

$$\left[\left(\int_{\frac{1}{2}}^1 \left|f'\left(sa+(1-s)b\right)\right|^p ds\right)^{\frac{1}{p}}+\left(\int_{\frac{1}{2}}^1 \left|f'\left((1-s)a+sb\right)\right|^p ds\right)^{\frac{1}{p}}\right]$$

$$\leq \frac{b-a}{4\,\Omega(\psi,\beta)}\left(2\int_0^1 \left|A_{\psi,\beta}(s)\right|^{p'} ds\right)^{\frac{1}{p'}}$$

$$\left[\int_0^1 \left|f'\left(sa+(1-s)b\right)\right|^p ds+\int_0^1 \left|f'\left((1-s)a+sb\right)\right|^p ds\right]^{\frac{1}{p}}$$

$$+\frac{b-a}{2}\left[\int_{\frac{1}{2}}^1 \left|f'\left(sa+(1-s)b\right)\right|^p ds+\int_0^1 \left|f'\left((1-s)a+sb\right)\right|^p ds\right]^{\frac{1}{p}}.$$

Given that $\left|f'\right|^p$ is a h-convex and h is B-function, we obtain

$$\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\beta)}\left[{}^{\psi}\mathfrak{J}^{\beta}_{b^-}F(a)+{}^{\psi}\mathfrak{J}^{\beta}_{a^+}F(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{b-a}{4\,\Omega(\psi,\beta)}\left(2\int_0^1 \left|A_{\psi,\beta}(s)\right|^{p'} ds\right)^{\frac{1}{p'}}$$

$$\left[\int_0^1 (h(s)+h(1-s))\left[\left|f'(a)\right|^p+\left|f'(b)\right|^p\right]ds\right]^{\frac{1}{p}}$$

$$+\frac{b-a}{2}\left[\int_{\frac{1}{2}}^1 (h(s)+h(1-s))\left[\left|f'(a)\right|^p+\left|f'(b)\right|^p\right]ds\right]^{\frac{1}{p}}$$

$$\leq \frac{b-a}{4\,\Omega(\psi,\beta)}\left(2\int_0^1 \left|A_{\psi,\beta}(s)\right|^{p'} ds\right)^{\frac{1}{p'}}\left[2\,h\left(\frac{1}{2}\right)\left[\left|f'(a)\right|^p+\left|f'(b)\right|^p\right]\right]^{\frac{1}{p}}$$

$$+\frac{b-a}{2}\left[h\left(\frac{1}{2}\right)\left[\left|f'(a)\right|^p+\left|f'(b)\right|^p\right]\right]^{\frac{1}{p}}$$

$$=\frac{(b-a)\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}}{2}$$

$$\left[1+\left(\frac{1}{\Omega^{p'}(\psi,\beta)}\int_0^1 \left|A_{\psi,\beta}(s)\right|^{p'} ds\right)^{\frac{1}{p'}}\right]\left[\left|f'(a)\right|^p+\left|f'(b)\right|^p\right]^{\frac{1}{p}}$$

$$\leq \frac{(b-a)\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}}{2}2^{1-\frac{1}{p'}}$$

$$\left(1+\frac{1}{\Omega^{p'}(\psi,\beta)}\int_0^1 \left|A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left[\left|f'(a)\right|^p+\left|f'(b)\right|^p\right]^{\frac{1}{p}}$$
$$=\frac{(b-a)\left(2h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}}{2}\left(1+\frac{1}{\Omega^{p'}(\psi,\beta)}\int_0^1 \left|A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left[\left|f'(a)\right|^p+\left|f'(b)\right|^p\right]^{\frac{1}{p}}.$$

This accomplishes the first inequality in (6.14).
Notice that for $p>1$ and $A, B\geq 0$, we have $A^p+B^p\leq (A+B)^p$, which yields to the second inequality in (6.14). □

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Data availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no competing interests.

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