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Generalized refinement of Hermite-Hadamard inequality

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Abstract

The purpose of this study is to develop the further refinement of Hermite-Hadamard-type inequalities. Following that, we will highlight, as a specific case, the recently obtained second Hermite-Hadamard-type inequalities, which are an improvement over the Hermite-Hadamard-type inequalities that are already in use.

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1 Introduction and preliminary

One of the most active areas of research in the fields of real and functional analysis is the study of convex integral inequalities, which are characterized by the presence of convex functions as important players. The establishment of limits for integrals, the exploration of function spaces, and the investigation of variational issues are all critical applications of these inequalities. In addition to this, they may be found in the theories of probability, optimization, and partial differential equations. Due to the fact that researchers are always discovering new applications and expanding upon previously discovered conclusions, these inequalities have evolved into vital resources for both theoretical and quantitative mathematics. As the relationship between convexity and integration becomes more complex, it paves the way for new areas of investigation and invention in a variety of mathematical fields.

For a convex function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ and $a, b \in I$ with $a < b$, where I is an interval of real numbers, C. Hermite and J. Hadamard established the following double inequality (H.H inequality) [1]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Inequalities in (1.1) are reversed if f is a concave function.

Tseng et al. [2] have recently established a Bullen-type inequality that refines the inequality (1.1). Let $f : [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$. Then, the following inequalities

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hold:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(t)dt \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{1.2}$$

For a convex function $f:[a,b]\to\mathbb{R}$, the subsequent inequalities are widely recognized in the literature as:

(1) Midpoint inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt.$$

In [3], Kirmaci proved the following estimation for functions characterized by the convexity of the absolute values of their first derivatives.

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{8}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

(2) Trapezoid inequality

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

In [4], Dragomir and Agarwal proved the following results:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{8}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{1.3}$$

(3) Bullen inequality.

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right].$$

In 2016, the authors provided the following estimation [5]:

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{16}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{1.4}$$

Throughout the study, we assume that $f:[a,b]\to\mathbb{R}$ is a convex function defined on the interval $[a,b]$ and $n\geq 1$ is an integer.

The general idea of this work is based on two principles:

- The first principle is to divide the interval $[a,b]$ into finite subintervals, verifying the two conditions:
  (1) All the subintervals have the same length.
  (2) The entire set of its subintervals is symmetric with respect to the midpoint $\frac{a+b}{2}$.
- The second principle is to apply Hermite-Hadamard inequalities to its subintervals.

In accordance with the first principle, the interval can be divided into even numbers of subintervals, yielding the geometric approach. We have the even numbers $2,4,8,16,\ldots,2^n,\ldots$.

2 Further refinement to the Hermite-Hadamard-type inequalities

In this part, we will discuss the fundamental lemmas that are necessary for the proof of the principal results.

Lemma 2.1 *Let $n \geq 1$ and $f:[a,b]\to\mathbb{R}$ be a convex function on the interval $[a,b]$. Ensure that the interval $[a,b]$ is divided into 2^n subintervals defined as $\left[\frac{(2^n-k+1)a+(k-1)b}{2^n}, \frac{(2^n-k)a+kb}{2^n}\right]$, then*

$$\begin{aligned}&\frac{1}{2^n}\sum_{k=1}^{2^n} f\left(\frac{(2^{n+1}-2k+1)a+(2k-1)b}{2^{n+1}}\right)\\ &\leq \frac{1}{b-a}\int_a^b f(t)dt \\ &\leq \frac{1}{2^n}\left[\sum_{k=1}^{2^n-1} f\left(\frac{(2^n-k)a+k\,b}{2^n}\right)+\frac{f(a)+f(b)}{2}\right].\end{aligned} \tag{2.1}$$

Proof By fixing the number $n\geq 1$ and the interval $[a,b]$, we create 2^n subintervals of the form $\left[\frac{(2^n-k+1)a+(k-1)b}{2^n}, \frac{(2^n-k)a+kb}{2^n}\right]$, these subintervals have the same length of $\frac{b-a}{2^n}$, with k taking values from 1 to 2^n. Using the Hermite-Hadamard inequalities (1.1) on the given subintervals produces

$$\begin{aligned}f\left(\frac{(2^{n+1}-1)a+b}{2^{n+1}}\right) &\leq \frac{2^n}{b-a}\int_a^{\frac{(2^n-1)a+b}{2^n}} f(t)dt\\ &\leq \frac{f(a)+f\left(\frac{(2^n-1)a+b}{2^n}\right)}{2},\end{aligned}$$

and

$$\begin{aligned}f\left(\frac{(2^{n+1}-3)a+3b}{2^{n+1}}\right) &\leq \frac{2^n}{b-a}\int_{\frac{(2^n-1)a+b}{2^n}}^{\frac{(2^n-2)a+2b}{2^n}} f(t)dt\\ &\leq \frac{f\left(\frac{(2^n-1)a+b}{2^n}\right)+f\left(\frac{(2^n-2)a+2b}{2^n}\right)}{2},\end{aligned}$$

$$\vdots$$

$$f\left(\frac{(2^{n+1}-2k+1)a+(2k-1)b}{2^{n+1}}\right) \leq \frac{2^n}{b-a}\int_{\frac{(2^n-k+1)a+(k-1)b}{2^n}}^{\frac{(2^n-k)a+kb}{2^n}} f(t)dt$$

$$\leq \frac{f\left(\frac{(2^n-k+1)a+(k-1)b}{2^n}\right)+f\left(\frac{(2^n-k)a+kb}{2^n}\right)}{2},$$

$$\vdots$$

$$f\left(\frac{3a+\left(2^{n+1}-3\right)b}{2^{n+1}}\right) \leq \frac{2^n}{b-a}\int_{\frac{2a+(2^n-2)b}{2^n}}^{\frac{a+(2^n-1)b}{2^n}} f(t)dt$$

$$\leq \frac{f\left(\frac{2a+(2^n-2)b}{2^n}\right)+f\left(\frac{a+(2^n-1)b}{2^n}\right)}{2},$$

and

$$f\left(\frac{a+\left(2^{n+1}-1\right)b}{2^{n+1}}\right) \leq \frac{2^n}{b-a}\int_{\frac{a+(2^n-1)b}{2^n}}^{b} f(t)dt$$

$$\leq \frac{f\left(\frac{a+(2^n-1)b}{2^n}\right)+f(b)}{2},$$

By summing the 2^n sides of inequalities, we obtain

$$\frac{1}{2^n}\sum_{k=1}^{2^n} f\left(\frac{(2^{n+1}-2k+1)a+(2k-1)b}{2^{n+1}}\right)$$
$$\leq \frac{1}{b-a}\int_a^b f(t)dt$$
$$\leq \frac{1}{2^n}\left[\sum_{k=1}^{2^n-1} f\left(\frac{(2^n-k)a+k\,b}{2^n}\right)+\frac{f(a)+f(b)}{2}\right].$$

□

Remark 1 Replacing n with $n-1$ yields the interval $[a,b]$, containing 2^{n-1} subintervals $\left[\frac{(2^{n-1}-k+1)a+(k-1)b}{2^{n-1}}, \frac{(2^{n-1}-k)a+kb}{2^{n-1}}\right]$ have the same length $\frac{b-a}{2^{n-1}}$, where $k = 1, 2, 3, \ldots, 2^{n-1}$.
Therefore

$$\frac{1}{2^{n-1}}\sum_{k=1}^{2^{n-1}} f\left(\frac{(2^n-2k+1)\,a+(2k-1)b}{2^n}\right)$$
$$\leq \frac{1}{b-a}\int_a^b f(t)dt \tag{2.2}$$
$$\leq \frac{1}{2^{n-1}}\left[\sum_{k=1}^{2^{n-1}-1} f\left(\frac{\left(2^{n-1}-k\right)a+k\,b}{2^{n-1}}\right)+\frac{f(a)+f(b)}{2}\right].$$

Lemma 2.2 *Let $n \geq 1$ and $f : [a,b] \to \mathbb{R}$ be a convex function on the interval $[a,b]$. Then*

$$\frac{1}{2^{n-1}}\sum_{k=1}^{2^{n-1}} f\left(\frac{(2^n-2k+1)\,a+(2k-1)b}{2^n}\right) \leq \frac{1}{2^n}\sum_{k=1}^{2^n} f\left(\frac{(2^{n+1}-2k+1)a+(2k-1)b}{2^{n+1}}\right). \tag{2.3}$$

Proof We have

$$\begin{aligned}&\frac{(2^n-2k+1)a+(2k-1)b}{2^n}\\&=\frac{(2^{n+1}-4k+2)a+(4k-2)b}{2^{n+1}}\\&=\frac{1}{2}\frac{\left[(2^{n+1}-4k+1)a+(4k-1)b\right]+\left[(2^{n+1}-4k+3)a+(4k-3)b\right]}{2^{n+1}}.\end{aligned}$$

Since f is a convex function, it follows that:

$$\begin{aligned}&f\left(\frac{(2^n-2k+1)a+(2k-1)b}{2^n}\right)\\&\leq\frac{1}{2}\left[f\left(\frac{(2^{n+1}-4k+1)a+(4k-1)b}{2^{n+1}}\right)+f\left(\frac{(2^{n+1}-4k+3)a+(4k-3)b}{2^{n+1}}\right)\right].\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{2^{n-1}}\sum_{k=1}^{2^{n-1}}f\left(\frac{(2^n-2k+1)a+(2k-1)b}{2^n}\right)\\&\leq\frac{1}{2^n}\sum_{k=1}^{2^{n-1}}\left[f\left(\frac{(2^{n+1}-4k+1)a+(4k-1)b}{2^{n+1}}\right)+f\left(\frac{(2^{n+1}-4k+3)a+(4k-3)b}{2^{n+1}}\right)\right].\end{aligned}\tag{2.4}$$

Taking $j=2k$ into the first sum in (2.4) yields

$$\sum_{k=1}^{2^{n-1}}f\left(\frac{(2^{n+1}-4k+1)a+(4k-1)b}{2^{n+1}}\right)=\sum_{j}f\left(\frac{(2^{n+1}-2j+1)a+(2j-1)b}{2^{n+1}}\right),\tag{2.5}$$

for $j=2,4,6,\ldots,2^n$.

Putting $j=(2k-1)$ into the second sum in (2.4) results

$$\sum_{k=1}^{2^{n-1}}f\left(\frac{(2^{n+1}-4k+3)a+(4k-3)b}{2^{n+1}}\right)=\sum_{j}f\left(\frac{(2^{n+1}-2j+1)a+(2j-1)b}{2^{n+1}}\right),\tag{2.6}$$

for $j=1,3,5,\ldots,(2^n-1)$.

Replacing (2.5) and (2.6) into (2.4) gives

$$\begin{aligned}&\frac{1}{2^{n-1}}\sum_{k=1}^{2^{n-1}}f\left(\frac{(2^n-2k+1)a+(2k-1)b}{2^n}\right)\\&\leq\frac{1}{2^n}\sum_{k=1}^{2^n}f\left(\frac{(2^{n+1}-2k+1)a+(2k-1)b}{2^{n+1}}\right).\end{aligned}$$

□

As a result of Lemma 2.1 and Lemma 2.2, we succeed in obtaining the following result.

Corollary 2.1 (Generalized refinement of midpoint inequalities) *Let $f:[a,b]\to\mathbb{R}$ be a convex function on the interval $[a,b]$. For all $n\geq 1$, we denote*

$$L_n=\frac{1}{2^n}\sum_{k=1}^{2^n}f\left(\frac{(2^{n+1}-2k+1)a+(2k-1)b}{2^{n+1}}\right)$$

and

$$L_{n-1}=\frac{1}{2^{n-1}}\sum_{k=1}^{2^{n-1}}f\left(\frac{(2^{n}-2k+1)a+(2k-1)b}{2^{n}}\right).$$

Consequently

$$L_0=f\left(\frac{a+b}{2}\right)\leq L_1\leq\cdots L_n\leq\cdots\leq\frac{1}{b-a}\int_a^b f(t)dt. \tag{2.7}$$

Lemma 2.3 *Let $n\geq 1$ and $f:[a,b]\to\mathbb{R}$ be a convex function on the interval $[a,b]$. Then*

$$\begin{aligned}&\frac{1}{2^n}\left[\sum_{k=1}^{2^n-1}f\left(\frac{(2^n-k)a+kb}{2^n}\right)+\frac{f(a)+f(b)}{2}\right]\\&\leq\frac{1}{2^{n-1}}\left[\sum_{k=1}^{2^{n-1}-1}f\left(\frac{(2^{n-1}-k)a+kb}{2^{n-1}}\right)+\frac{f(a)+f(b)}{2}\right].\end{aligned} \tag{2.8}$$

Proof The inequality (2.8) can be rewritten as follows:

$$\sum_{k=1}^{2^n-1}f\left(\frac{(2^n-k)a+kb}{2^n}\right)\leq 2\sum_{k=1}^{2^{n-1}-1}f\left(\frac{(2^{n-1}-k)a+kb}{2^{n-1}}\right)+\frac{f(a)+f(b)}{2}. \tag{2.9}$$

It is sufficient to demonstrate the previously mentioned inequality (2.9). By altering the index k, we obtain

$$\begin{aligned}&\sum_{k=1}^{2^n-1}f\left(\frac{(2^n-k)a+kb}{2^n}\right)\\&=\sum_{j=1}^{2^{n-1}-1}f\left(\frac{(2^n-(2j))a+(2j)b}{2^n}\right)+\sum_{j=1}^{2^{n-1}}f\left(\frac{(2^n-(2j-1))a+(2j-1)b}{2^n}\right)\\&=\sum_{j=1}^{2^{n-1}-1}f\left(\frac{(2^{n-1}-j)a+jb}{2^{n-1}}\right)\\&\quad+\sum_{j=1}^{2^{n-1}}f\left(\frac{1}{2}\left[\frac{[(2^{n-1}-j)a+jb]+[(2^{n-1}-(j-1))a+(j-1)b]}{2^{n-1}}\right]\right).\end{aligned}$$

Using the convexity of the function f results

$$\sum_{k=1}^{2^n-1}f\left(\frac{(2^n-k)a+kb}{2^n}\right)$$

$$\leq \sum_{j=1}^{2^{n-1}-1} f\left(\frac{\left(2^{n-1}-j\right)a+jb}{2^{n-1}}\right)$$
$$+\frac{1}{2}\left[\sum_{j=1}^{2^{n-1}} f\left(\frac{\left(2^{n-1}-j\right)a+jb}{2^{n-1}}\right)+\sum_{j=1}^{2^{n-1}} f\left(\frac{\left(2^{n-1}-(j-1)\right)a+(j-1)b}{2^{n-1}}\right)\right].$$

Then, we have

$$\sum_{k=1}^{2^n-1} f\left(\frac{(2^n-k)a+kb}{2^n}\right)$$
$$\leq \sum_{j=1}^{2^{n-1}-1} f\left(\frac{\left(2^{n-1}-j\right)a+jb}{2^{n-1}}\right)$$
$$+\frac{1}{2}\left[\sum_{j=1}^{2^{n-1}-1} f\left(\frac{\left(2^{n-1}-j\right)a+jb}{2^{n-1}}\right)+f(b)+\sum_{i=0}^{2^{n-1}-1} f\left(\frac{\left(2^{n-1}-i\right)a+ib}{2^{n-1}}\right)\right].$$

Hence

$$\sum_{k=1}^{2^n-1} f\left(\frac{(2^n-k)a+kb}{2^n}\right)$$
$$\leq \sum_{j=1}^{2^{n-1}-1} f\left(\frac{\left(2^{n-1}-j\right)a+jb}{2^{n-1}}\right)$$
$$+\frac{1}{2}\left[\sum_{j=1}^{2^{n-1}-1} f\left(\frac{\left(2^{n-1}-j\right)a+jb}{2^{n-1}}\right)+f(b)+\sum_{i=1}^{2^{n-1}-1} f\left(\frac{\left(2^{n-1}-i\right)a+ib}{2^{n-1}}\right)+f(a)\right]$$
$$=2\sum_{k=1}^{2^{n-1}-1} f\left(\frac{(2^{n-1}-k)a+kb}{2^{n-1}}\right)+\frac{f(a)+f(b)}{2}.$$

This establishes the desired inequality (2.9), which confirms the inequality (2.8). □

Based on Lemma 2.1 and Lemma 2.3, we achieve the following result.

Corollary 2.2 (Generalized refinement of trapezoid inequalities) *Let $f:[a,b]\to\mathbb{R}$ be a convex function on the interval $[a,b]$. For all $n\geq 1$, we denote*

$$R_n=\frac{1}{2^n}\left[\sum_{k=1}^{2^n-1} f\left(\frac{(2^n-k)a+kb}{2^n}\right)+\frac{f(a)+f(b)}{2}\right],$$

and

$$R_{n-1}=\frac{1}{2^{n-1}}\left[\sum_{k=1}^{2^{n-1}-1} f\left(\frac{\left(2^{n-1}-k\right)a+kb}{2^{n-1}}\right)+\frac{f(a)+f(b)}{2}\right].$$

In fact, the expression $\sum_{k=1}^{0} f\left(\frac{(2^{n-1}-k)a+kb}{2^{n-1}}\right) = 0$ *holds true, consequently*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \cdots \leq R_n \leq \cdots \leq R_1 \leq R_0 = \frac{f(a)+f(b)}{2}. \tag{2.10}$$

The nth refinement of Hermite-Hadamard-type inequalities and the $(n-1)$th refinement of Bullen-type inequalities are now presented.

Theorem 2.1 *Let* $n \geq 1$ *be a fixed integer and* $f : [a,b] \to \mathbb{R}$ *be a convex function on the interval* $[a,b]$. *Ensure that the interval* $[a,b]$ *is divided into* 2^n *subintervals defined as* $\left[\frac{(2^n-k+1)a+(k-1)b}{2^n}, \frac{(2^n-k)a+kb}{2^n}\right]$, *then*

$$\begin{aligned}
&\frac{1}{2^{n-1}}\sum_{k=1}^{2^{n-1}} f\left(\frac{(2^n-2k+1)\,a+(2k-1)b}{2^n}\right)\\
&\leq \frac{1}{2^n}\sum_{k=1}^{2^n} f\left(\frac{(2^{n+1}-2k+1)a+(2k-1)b}{2^{n+1}}\right)\\
&\leq \frac{1}{b-a}\int_a^b f(t)dt\\
&\leq \frac{1}{2^n}\left[\sum_{k=1}^{2^n-1} f\left(\frac{(2^n-k)a+k\,b}{2^n}\right)+\frac{f(a)+f(b)}{2}\right]\\
&\leq \frac{1}{2^{n-1}}\left[\sum_{k=1}^{2^{n-1}-1} f\left(\frac{\left(2^{n-1}-k\right)a+k\,b}{2^{n-1}}\right)+\frac{f(a)+f(b)}{2}\right].
\end{aligned} \tag{2.11}$$

Proof The desired inequality (2.11) is derived by amalgamating the preceding inequalities, (2.2), (2.3), and (2.8). □

Remark 2 Replacing $n = 1$ into Theorem 2.1 yields the resulting Hermite-Hadamard-type inequalities (1.2).

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\\
&\leq \frac{1}{b-a}\int_a^b f(t)dt\\
&\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}.
\end{aligned}$$

The second refinement of Hermite-Hadamard-type inequalities is derived from the inclusion of $n = 2$ in Theorem 2.1, which represents a novel enhancement to Hermite-Hadamard-type inequalities.

Corollary 2.3 *Let* $f : [a,b] \to \mathbb{R}$ *be a convex function on* $[a,b]$. *Then, the following second Hermite-Hadamard-type inequalities hold*:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]$$

$$\begin{aligned}
&\leq \frac{1}{4}\left[f\left(\frac{7a+b}{8}\right)+f\left(\frac{5a+3b}{8}\right)+f\left(\frac{3a+5b}{8}\right)+f\left(\frac{a+7b}{8}\right)\right] \\
&\leq \frac{1}{b-a}\int_a^b f(t)dt \qquad (2.12)\\
&\leq \frac{1}{4}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)+f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \\
&\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\leq \frac{f(a)+f(b)}{2}.
\end{aligned}$$

The following result is derived from the above Corollary 2.1 and Corollary 2.2, which represent a novel enhancement to Hermite-Hadamard-type inequalities.

Corollary 2.4 (Generalized refinement of Hermite-Hadamard inequality) *Let* $f:[a,b]\to\mathbb{R}$ *be a convex function on* $[a,b]$. *For all* $n\geq 1$, *we denote*

$$L_n=\frac{1}{2^n}\sum_{k=1}^{2^n} f\left(\frac{(2^{n+1}-2k+1)a+(2k-1)b}{2^{n+1}}\right),$$

and

$$R_n=\frac{1}{2^n}\left[\sum_{k=1}^{2^n-1} f\left(\frac{(2^n-k)a+k\,b}{2^n}\right)+\frac{f(a)+f(b)}{2}\right].$$

Then, the following generalized refinement of Hermite-Hadamard inequalities holds:

$$f\left(\frac{a+b}{2}\right)\leq L_1\leq\cdots L_n\leq\cdots\leq\frac{1}{b-a}\int_a^b f(t)dt\leq\cdots\leq R_n\leq\cdots\leq R_1\leq\frac{f(a)+f(b)}{2}. \qquad (2.13)$$

3 Applications

We present here some examples of simple convex functions on the interval $[0,2]$. (see Figure 1)

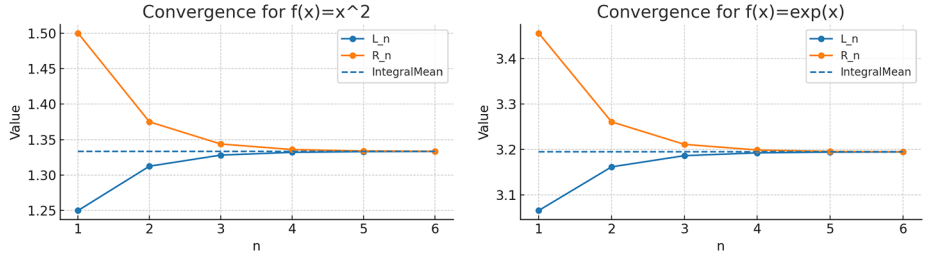

Figure 1 Convergence of L_n and R_n versus n with the integral means

Author contributions
All the authors contributed equally to this study, discussed the mathematical results, validated them and confirmed the last version.

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