Iran J Sci Technol Trans Sci (2020) 44:425–435
https://doi.org/10.1007/s40995-020-00843-1

RESEARCH PAPER

Powers Sums of the First and Second Kinds of Chebyshev Polynomials

Emrah Kılıç[1] · Sibel Koparal[2] · Neşe Ömür[2]

Received: 21 January 2019 / Accepted: 24 February 2020 / Published online: 9 April 2020
© Shiraz University 2020

Abstract
Odd powers of even-indexed Chebyshev polynomials of the second kind and odd powers of odd-indexed Chebyshev polynomials of the first kind were computed. In this paper, we evaluate all of the rest kinds of power sums of the Chebyshev polynomials. We present the relationships between the Chebyshev polynomials and general Fibonacci, Lucas sequences.

Keywords The first and second kinds Chebyshev polynomials · Power sums · Alternating sums · Fibonacci and Lucas numbers

Mathematics Subject Classification 05A10 · 11B37

1 Introduction

The Chebyshev polynomials of the first and second kinds are defined as follows for $n > 0$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$
$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$$

with $T_0(x) = 1$, $T_1(x) = x$ and $U_0(x) = 1, U_1(x) = 2x$, respectively.

The Binet formulae of $T_n(x)$ and $U_n(x)$ are

$$T_n(x) = \frac{1}{2}\left[\left(x + \sqrt{x^2-1}\right)^n + \left(x - \sqrt{x^2-1}\right)^n\right] \quad (1)$$

and

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left[\left(x + \sqrt{x^2-1}\right)^{n+1} - \left(x - \sqrt{x^2-1}\right)^{n+1}\right]. \quad (2)$$

The general Fibonacci sequence $\{u_n(A,B)\}$ and general Lucas sequence $\{v_n(A,B)\}$ are defined as for $n > 0$

$$u_{n+1}(A,B) = Au_n(A,B) + Bu_{n-1}(A,B)$$

and

$$v_{n+1}(A,B) = Av_n(A,B) + Bv_{n-1}(A,B)$$

with $u_0(A,B) = 0, u_1(A,B) = 1$ and $v_0(A,B) = 2$, $v_1(A,B) = A$, respectively. We frequently denote these sequences briefly as $\{u_n\}$ and $\{v_n\}$.

The Binet formulae are

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } v_n = \alpha^n + \beta^n,$$

where $\alpha, \beta = \left(A \pm \sqrt{A^2 + 4B}\right)/2$.

Especially, if $A = B = 1$, then $u_n = F_n$ (nth Fibonacci number) and $v_n = L_n$ (nth Lucas number).

Prodinger (2008) computed the sums

$$\sum_{k=0}^{n} F_{2k+\delta}^{2m+\varepsilon}$$

and their Lucas counterparts with $\varepsilon, \delta \in \{0, 1\}$. He showed that

✉ Sibel Koparal
sibel.koparal@kocaeli.edu.tr

Emrah Kılıç
ekilic@etu.edu.tr

Neşe Ömür
neseomur@kocaeli.edu.tr

[1] Department of Mathematics, TOBB University of Economics and Technology, 06560 Ankara, Turkey

[2] Department of Mathematics, Kocaeli University, 41380 Izmit, Kocaeli, Turkey

$$\sum_{k=0}^{n} L_{2k}^{2m+1} = \sum_{j=0}^{m} \binom{2m+1}{j} \frac{L_{(2m+1-2j)(2n+1)}}{L_{2m+1-2j}} + 4^m$$

and

$$\sum_{k=0}^{n} L_{2k}^{2m} = \sum_{j=0}^{m-1} \binom{2m}{j} \frac{F_{(2m-2j)(2n+1)}}{F_{2m-2j}} + \binom{2m}{m}\left(n+\frac{1}{2}\right) + 2^{2m-1}.$$

Kılıç et al. (2011) computed the following sums as alternating analogues of the sums given in Prodinger (2008)

$$\sum_{k=1}^{n} (-1)^k F_{2k+\delta}^{2m+\varepsilon} \quad \text{and} \quad \sum_{k=1}^{n} (-1)^k L_{2k+\delta}^{2m+\varepsilon},$$

where $\varepsilon, \delta \in \{0, 1\}$. They showed that for $n > 0$ and positive odd m,

$$\sum_{k=1}^{n} (-1)^k F_k^{2m} = \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{i(n+1)+n} \binom{2m}{i} \frac{F_{(m-i)(2n+1)}}{F_{m-i}} - \frac{1}{5^m}\binom{2m-1}{m} - \frac{1}{5^m}\binom{2m}{m} n,$$

and, for $m > 0$ and even $n > 0$

$$\sum_{k=1}^{n} (-1)^k L_{2k}^{2m} = \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} + \binom{2m-1}{m} - 2^{2m-1}.$$

Li (2015) obtained the following identities involving power sums of the Chebyshev polynomials $T_n(x)$ and $U_n(x)$: For any positive integers n and m,

$$\sum_{k=0}^{n} T_{2k+1}^{2m+1}(x) = \frac{1}{2^{2m+1}} \sum_{k=0}^{m} \binom{2m+1}{m-k} \frac{U_{2(2k+1)(n+1)-1}(x)}{U_{2k}(x)}$$

and

$$\sum_{k=0}^{n} U_{2k}^{2m+1}(x) = \frac{1}{4^m (x^2-1)^m} \times \sum_{k=0}^{m} \binom{2m+1}{m-k} (-1)^{m-k} \frac{U_{(2k+1)(n+1)-1}^2(x)}{U_{2k}(x)}.$$

Ma and Zhang (2007) obtained the common formulas with x^n being represented by Chebyshev polynomials, then get two identities about the Fibonacci numbers and Lucas numbers. They showed that for any nonnegative integer n and positive integer m,

$$L_n^{2m} = (-1)^{mn} \frac{(2m)!}{(m!)^2} + (2m)! \sum_{k=1}^{m} \frac{(-1)^{n(m-k)}}{(m-k)!(m+k)!} L_{2kn}$$

and

$$L_n^{2m+1} = \frac{2(2m+2)!}{F_n} \sum_{k=1}^{m} \frac{(-1)^{n(m-k)}(k+1)}{(m-k)!(m+k+2)!} F_{2(k+1)n}.$$

Wang and Zhang (2015) gave several interesting identities involving the derivative of the Chebyshev polynomials of the first kind as follows:

$$\sum_{k=1}^{n} \left(T_k'(x)\right)^2 = \frac{(-1)^m}{4^m (x^2-1)^m} \frac{(2m)!}{(n!)^2} \sum_{k=1}^{n} k^{2m} + \sum_{i=1}^{m+1} \sum_{k=1}^{m} \binom{2m}{m-k} (-1)^{m-k} \frac{r(i,n) T_{2ki}(x) + s(i,h) T_{2k(i-1)}(x)}{4^{m+i}(x^2-1)^{m+i} U_{k-1}^{2i}(x)} + \sum_{i=1}^{m+1} \sum_{k=1}^{m} \binom{2m}{m-k} (-1)^{m-k} \frac{p(i,n) T_{2k(n+i)}(x) + q(i,h) T_{2k(n+i-1)}(x)}{4^{m+i}(x^2-1)^{m+i} U_{k-1}^{2i}(x)},$$

where $r(i, n)$, $s(i, n)$, $p(i, n)$ and $q(i, n)$ are computable constants.

Zhang and Wang (2017) gave interesting results involving the integral of the power sums of the first kind Chebyshev polynomials. They showed that for positive integers n and m,

$$\sum_{k=0}^{\infty} \int_0^1 T_{2k}^{2m+1}(y) \mathrm{d}y = -\frac{1}{2} + \frac{\pi}{4^{m+1}} \sum_{k=0}^{m} \binom{2m+1}{m-k} \frac{1}{2k+1} \cot \frac{\pi}{4k+2}$$

and especially

$$\sum_{k=0}^{n} \int_0^x T_{2k}(y) \mathrm{d}y = \frac{T_{2n+1}(x)}{2(2n+1)} - \frac{x}{2}.$$

As we mentioned above, odd powers of even-indexed Chebyshev polynomials of the second kind and odd powers of odd-indexed Chebyshev polynomials of the first kind were computed by Li (2015). In this paper, we evaluate all of the rest kinds of power sums of the Chebyshev polynomials. We present the relationships between the Chebyshev polynomials and general Fibonacci, Lucas sequences.

2 The Chebyshev Polynomials, the General Fibonacci and Lucas Numbers

Now we will present relationships between the Chebyshev polynomials and general Fibonacci and Lucas numbers. First, we recall the following identities from Kılıç and Stănică (2009, 2011) that

$$u_{(n+1)k} = v_k u_{nk} - (-B)^k u_{(n-1)k}$$

and

$$v_{(n+1)k} = v_k v_{nk} - (-B)^k v_{(n-1)k}.$$

Define the tridiagonal matrices $H_n(x)$ and $M_n(x)$ of order n as

$$H_n(x) = \begin{bmatrix} 2x & 1 & & & \\ 1 & 2x & 1 & & \\ & 1 & 2x & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 2x \end{bmatrix}$$

and

$$M_n(x) = \begin{bmatrix} x & 1 & & & \\ 1 & 2x & 1 & & \\ & 1 & 2x & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 2x \end{bmatrix}.$$

We also note the following known equalities

$$\det H_n(x) = U_n(x) \quad \text{and} \quad \det M_n(x) = T_n(x),$$

where $T_n(x)$ and $U_n(x)$ stand for the Chebyshev polynomials of the first and second kinds, respectively.

We define the tridiagonal matrices $\check{H}_n$ and $\check{M}_n$ of order n as

$$\check{H}_n = \begin{bmatrix} v_k & \sqrt{(-B)^k} & & & \\ \sqrt{(-B)^k} & v_k & \sqrt{(-B)^k} & & \\ & \sqrt{(-B)^k} & v_k & \ddots & \\ & & \ddots & \ddots & \sqrt{(-B)^k} \\ & & & \sqrt{(-B)^k} & v_k \end{bmatrix}$$

and

$$\check{M}_n = \begin{bmatrix} v_k/2 & \sqrt{(-B)^k} & & & \\ \sqrt{(-B)^k} & v_k & \sqrt{(-B)^k} & & \\ & \sqrt{(-B)^k} & v_k & \ddots & \\ & & \ddots & \ddots & \sqrt{(-B)^k} \\ & & & \sqrt{(-B)^k} & v_k \end{bmatrix}.$$

From using the results of Kılıç and Stănică (2011), we note that

$$\det \check{H}_n = \frac{u_{k(n+1)}}{u_k} \quad \text{and} \quad \det \check{M}_n = \frac{1}{2} v_{kn},$$

where u_n and v_n stand for the general Fibonacci and Lucas numbers, respectively.

We give some relationships between the Chebyshev polynomials, the general Fibonacci and Lucas sequences. Consider

$$\begin{aligned} \frac{u_{k(n+1)}}{u_k} &= \det \check{H}_n \\ &= \begin{vmatrix} v_k & \sqrt{(-B)^k} & & & \\ \sqrt{(-B)^k} & v_k & \sqrt{(-B)^k} & & \\ & \sqrt{(-B)^k} & v_k & \ddots & \\ & & \ddots & \ddots & \sqrt{(-B)^k} \\ & & & \sqrt{(-B)^k} & v_k \end{vmatrix} \\ &= (-B)^{nk/2} \begin{vmatrix} v_k/(-B)^{k/2} & 1 & & & \\ 1 & v_k/(-B)^{k/2} & 1 & & \\ & 1 & v_k/(-B)^{k/2} & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & v_k/(-B)^{k/2} \end{vmatrix} \\ &= (-B)^{nk/2} \det H_n\left(\frac{v_k}{2(-B)^{k/2}}\right), \end{aligned}$$

which gives us the relation between the Chebyshev polynomial of the second kind, $U_n(x)$, and the general Fibonacci numbers, u_n of the form

$$(-B)^{nk/2} U_n\left(\frac{v_k}{2(-B)^{k/2}}\right) = \frac{u_{k(n+1)}}{u_k}.$$

We consider a special case of the last equality. If we take $B = 1$ in the definition of $\{u_n(A, B)\}$ and $k = 2$, it takes the form

$$(-1)^n U_n\left(\frac{v_2(A,1)}{-2}\right) = \frac{u_{2(n+1)}(A,1)}{u_2(A,1)}$$

or

$$U_n\left(\frac{v_2(A,1)}{2}\right) = \frac{u_{2(n+1)}(A,1)}{u_2(A,1)}.$$

If $A = 1$, then $u_n(1,1) = F_n$(nth Fibonacci number) and $v_n(1,1) = L_n$(nth Lucas number) and so we derive the result

$$U_n\left(\frac{3}{2}\right) = F_{2(n+1)}.$$

A similar relation between the Chebyshev polynomial of the first kind and general Lucas number could be given. For this consider

$$\frac{1}{2}v_{kn} = \det \check{M}_n$$

$$= \begin{vmatrix} v_k/2 & \sqrt{(-B)^k} & & & \\ \sqrt{(-B)^k} & v_k & \sqrt{(-B)^k} & & \\ & \sqrt{(-B)^k} & v_k & \ddots & \\ & & \ddots & \ddots & \sqrt{(-B)^k} \\ & & & \sqrt{(-B)^k} & v_k \end{vmatrix}$$

$$= (-B)^{nk/2} \begin{vmatrix} v_k/2(-B)^{k/2} & 1 & & & \\ 1 & v_k/(-B)^{k/2} & 1 & & \\ & 1 & v_k/(-B)^{k/2} & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & v_k/(-B)^{k/2} \end{vmatrix}$$

$$= (-B)^{nk/2} \det M_n\left(\frac{v_k}{2(-B)^{k/2}}\right),$$

which gives us the relation between the Chebyshev polynomials of the first kind $T_n(x)$ and the general Lucas number v_n of the form

$$(-B)^{nk/2} T_n\left(\frac{v_k}{2(-B)^{k/2}}\right) = \frac{1}{2}v_{kn}.$$

If we take $B = 1$ in the definition of $\{v_n(A, B)\}$ and $k = 2$, the last equality takes the form

$$(-1)^n T_n\left(\frac{v_2(A,1)}{-2}\right) = \frac{v_{2n}(A,1)}{2}.$$

Also we can rearrange it as

$$T_n\left(\frac{v_2(A,1)}{2}\right) = \frac{v_{2n}(A,1)}{2}.$$

If we take $A = 1$, then $v_n(1, 1) = L_n$(nth Lucas number) and so we derive the specific result

$$T_n\left(\frac{3}{2}\right) = \frac{1}{2}L_{2n}.$$

3 Alternating Sums of the Chebyshev Polynomials

In this section, we will start with some auxiliary lemmas and then present our main results about power sums of the Chebyshev polynomials of first and second kinds and their alternating analogues.

Lemma 1 *For nonzero integer a, any integer b and for $n > 0$,*

$$\text{(i)} \sum_{k=1}^{n} P_{ak+b}(x) = \frac{P_{a+b}(x) - P_b(x) - P_{a(n+1)+b}(x) + P_{an+b}(x)}{2(1 - P_a(x))},$$

and

$$\text{(ii)} \sum_{k=1}^{n} (-1)^k P_{ak+b}(x) = \frac{(-1)^n[-P_{a(n+1)+b}(x) + P_{an+b}(x)] - P_{a+b}(x) - P_b(x)}{2(1 + P_a(x))},$$

where $P_n(x)$ stands for both the Chebyshev polynomials $T_n(x)$ or $U_n(x)$.

Proof By the Binet formulae of $T_n(x)$ and $U_n(x)$, the claims follow.

□

Lemma 2 *For any integers a and b,*

(i) $T_{a+b}(x) + T_{a-b}(x) = 2T_a(x)T_b(x)$,
(ii) $T_{a+b}(x) - T_{a-b}(x) = 2(x^2 - 1)U_{a-1}(x)U_{b-1}(x)$,
(iii) $U_{a+b}(x) + U_{a-b}(x) = 2U_a(x)T_b(x)$,
(iv) $U_{a+b}(x) - U_{a-b}(x) = 2T_{a+1}(x)U_{b-1}(x)$,
(v) $T_{2(a-b)}(x) + 1 = 2T^2_{a-b}(x)$,
(vi) $T_{2(a-b)}(x) - 1 = 2\left(x^2 - 1\right)U^2_{a-b-1}(x)$.

Proof By the Binet formulae of $T_n(x)$ and $U_n(x)$, the claims easily follow.

□

Lemma 3 *For $n > 0$, we have*

$$\sum_{i=0}^{n-1} \binom{2n}{i} v_{k(2n-2i)}(-B)^{ki} = v_k^{2n} - \binom{2n}{n}(-B)^{kn},$$

$$\sum_{i=0}^{n-1} \binom{2n}{i}(-1)^i v_{k(2n-2i)}(-B)^{ki} = \left[v_k^2 - 4(-B)^k\right]^n - (-1)^n \binom{2n}{n}(-B)^{kn},$$

$$\sum_{i=0}^{n-1} \binom{2n}{i}(-1)^i v_{k(4n-4i)}B^{2ki} = \left[v_k^2 - 4(-B)^k\right]^n v_k^{2n} - (-1)^n \binom{2n}{n}B^{2kn},$$

$$\sum_{i=0}^{n} \binom{2n+1}{i}(-1)^i u_{k(2n-2i+1)}(-B)^{ki} = \left[v_k^2 - 4(-B)^k\right]^n u_k. \tag{3}$$

Proof Clearly, by the Binet formula, we see that

$$\sum_{k=0}^{2n}\binom{2n}{k}\frac{v_{k(2n-2i)}}{(-B)^{-ki}}=2\left(v_k^{2n}\right)$$

and so

$$\sum_{k=0}^{2n}\binom{2n}{k}\frac{v_{k(2n-2i)}}{(-B)^{-ki}}=\sum_{k=0}^{n-1}\binom{2n}{k}\frac{v_{k(2n-2i)}}{(-B)^{-ki}}+\frac{2}{(-B)^{-kn}}\binom{2n}{n}+\sum_{k=n+1}^{2n}\binom{2n}{k}\frac{v_{k(2n-2i)}}{(-B)^{-ki}}.$$

Denote

$$S_1=\sum_{k=0}^{n-1}\binom{2n}{k}\frac{v_{k(2n-2i)}}{(-B)^{-ki}}\quad\text{and}\quad S_2=\sum_{k=n+1}^{2n}\binom{2n}{k}\frac{v_{k(2n-2i)}}{(-B)^{-ki}}.$$

Consider by $v_{-kn}=(-B)^{-kn}v_{kn}$,

$$\begin{aligned}S_2&=\sum_{i=n+1}^{2n}\binom{2n}{k}\frac{v_{k(2n-2i)}}{(-B)^{-ki}}\\&=\sum_{n+1\le i+n+1\le 2n}\binom{2n}{n+i+1}\frac{v_{k(2n-2(n+i+1))}}{(-B)^{-k(n+i+1)}}\\&=\sum_{i=0}^{n-1}\binom{2n}{n+i+1}\frac{v_{-k(2i+2)}}{(-B)^{-k(n+i+1)}}\\&=\sum_{i=0}^{n-1}\binom{2n}{2n-i}\frac{v_{-k(2n-2i)}}{(-B)^{-k(2n-i)}}\\&=\sum_{i=0}^{n-1}\binom{2n}{i}\frac{v_{k(2n-2i)}}{(-B)^{-ki}}=S_1.\end{aligned}$$

Thus, we have

$$2v_k^{2n}=2S_1+\frac{2}{(-B)^{-kn}}\binom{2n}{n}$$

and so

$$S_1=\sum_{i=0}^{n-1}\binom{2n}{i}\frac{v_{k(2n-2i)}}{(-B)^{-ki}}=v_k^{2n}-(-B)^{kn}\binom{2n}{n},$$

as claimed. Similarly, the other results could be proven. □

Theorem 1 *For $m,n>0$,*

(i)
$$\sum_{k=1}^{n}T_{2k}^{2m}(x)=\frac{1}{2^{2m-1}}\sum_{i=0}^{m-1}\binom{2m}{i}\times\frac{U_{(2m-2i)n-1}(x)T_{(2m-2i)(n+1)}(x)}{U_{2m-2i-1}(x)}+\frac{n}{2^{2m}}\binom{2m}{m}.$$

(ii) For $x\in\mathbb{R}-\{-1,1\}$,

$$\sum_{k=1}^{n}U_{2k}^{2m}(x)=\frac{1}{2^{2m}(x^2-1)^m}\times\left[2\sum_{i=0}^{m-1}(-1)^i\binom{2m}{i}\frac{T_{(2m-2i)(n+2)}(x)U_{(2m-2i)n-1}(x)}{U_{2m-2i-1}(x)}+n(-1)^m\binom{2m}{m}\right].$$

Proof Consider

$$\begin{aligned}&\sum_{k=1}^{n}T_{2k}^{2m}(x)\\&=\sum_{k=1}^{n}\left[\frac{1}{2}\left(\left(x+\sqrt{x^2-1}\right)^{2k}+\left(x-\sqrt{x^2-1}\right)^{2k}\right)\right]^{2m}\\&=\sum_{k=1}^{n}\frac{1}{2^{2m}}\sum_{i=0}^{2m}\binom{2m}{i}\left(x+\sqrt{x^2-1}\right)^{2ki}\left(x-\sqrt{x^2-1}\right)^{2k(2m-i)}\\&=\sum_{k=1}^{n}\frac{1}{2^{2m}}\sum_{i=0}^{m}\binom{2m}{i}\left(\left(x+\sqrt{x^2-1}\right)^{2k(2m-i)}\left(x-\sqrt{x^2-1}\right)^{2ki}+\left(x+\sqrt{x^2-1}\right)^{2ki}\left(x-\sqrt{x^2-1}\right)^{2k(2m-i)}\right)\\&=\sum_{k=1}^{n}\frac{1}{2^{2m}}\sum_{i=0}^{m}\binom{2m}{i}\left(\left(x+\sqrt{x^2-1}\right)^{4k(m-i)}+\left(x-\sqrt{x^2-1}\right)^{4k(m-i)}\right)-\frac{n}{2^{2m}}\binom{2m}{m}\\&=\sum_{k=1}^{n}\frac{1}{2^{2m-1}}\sum_{i=0}^{m}\binom{2m}{i}T_{4k(m-i)}(x)-\frac{n}{2^{2m}}\binom{2m}{m}\\&=\frac{1}{2^{2m-1}}\sum_{k=1}^{m}\binom{2m}{i}\sum_{i=0}^{n}T_{4k(m-i)}(x)-\frac{n}{2^{2m}}\binom{2m}{m}\\&=\frac{1}{2^{2m-1}}\sum_{i=0}^{m-1}\binom{2m}{i}\sum_{k=1}^{n}T_{4k(m-i)}(x)+\frac{n}{2^{2m}}\binom{2m}{m}.\end{aligned}$$

If we take $a=4(m-i)$ and $b=0$ in Lemma 1, then we write

$$\sum_{k=1}^{n}T_{2k}^{2m}(x)=\frac{n}{2^{2m}}\binom{2m}{m}+\frac{1}{2^{2m-1}}\sum_{i=1}^{m}\binom{2m}{i}\times\frac{T_{4(m-i)}(x)-1-T_{4(m-i)(n+1)}(x)+T_{4(m-i)n}(x)}{2-2T_{4(m-i)}(x)}.$$

Using (ii), (iii) and (vi) in Lemma 2, we get

$$\sum_{k=1}^{n}T_{2k}^{2m}(x)=\frac{n}{2^{2m}}\binom{2m}{m}+\frac{1}{2^{2m-1}}\sum_{i=0}^{m-1}\binom{2m}{i}\times\frac{U_{(2m-2i)n-1}(x)T_{(2m-2i)(n+1)}(x)}{U_{2m-2i-1}(x)},$$

as claimed. □

Corollary 1 *For $m,n>0$,*

$$\sum_{j=1}^{n}\frac{v_{2kj}^{2m}}{B^{2kmj}}=\sum_{i=0}^{m-1}\binom{2m}{i}\frac{u_{k(2n+1)(2m-2i)}}{B^{2kn(m-i)}u_{k(2m-2i)}}-2^{2m-1}+\frac{2n+1}{2}\binom{2m}{m}$$

and

$$\sum_{j=0}^{n}\frac{u_{k(2j+1)}^{2m}}{B^{2kmj}}=\frac{u_k^{2m}}{\left(v_k^2-4(-B)^k\right)^m}\left(\sum_{i=0}^{m-1}(-1)^i\binom{2m}{i}\times\frac{u_{k(n+1)(4m-4i)}}{(-B)^{k(2n(m-i)-i)}u_{k(2m-2i)}}+(-B)^{km}(-1)^m(n+1)\binom{2m}{m}\right).$$

Proof Using $T_n\left(\frac{v_k}{2(-B)^{k/2}}\right)=\frac{v_{kn}}{2(-B)^{kn/2}}$ and $U_n\left(\frac{v_k}{2(-B)^{k/2}}\right)=\frac{1}{(-B)^{kn/2}}\frac{u_{k(2n+2)}}{u_k}$ in Theorem 1, we have

$$\sum_{j=1}^{n}\frac{v_{2kj}^{2m}}{(-B)^{2kmj}}=\sum_{i=0}^{m-1}\binom{2m}{i}\frac{u_{2k(m-i)n}v_{k(2m-2i)(n+1)}}{(-B)^{2kn(m-i)}u_{k(2m-2i)}}+n\binom{2m}{m}.$$

From the equality

$$u_{kt}v_{kn}=u_{k(n+t)}-(-B)^{kt}u_{k(n-t)},$$

and the sum

$$\sum_{i=0}^{n}\binom{2n}{i}=2^{2n-2}+2^{-1}\binom{2n}{n},$$

we write

$$\sum_{j=1}^{n}\frac{v_{2kj}^{2m}}{(-B)^{2kmj}}=\sum_{i=0}^{m-1}\binom{2m}{i}\frac{u_{k(2n+1)(2m-2i)}}{B^{2kn(m-i)}u_{k(2m-2i)}}-2^{2m-1}+\frac{2n+1}{2}\binom{2m}{m}.$$

The proof is complete. Similarly, the other desired result is obtained. □

Theorem 2 *For $n,m>0$,*

(i)

$$\sum_{k=1}^{n}T_{2k+1}^{2m}(x)=\frac{1}{2^{2m-1}}\sum_{i=0}^{m-1}\binom{2m}{i}\times\frac{T_{(2m-2i)(n+2)}(x)U_{(2m-2i)n-1}(x)}{U_{2m-2i-1}(x)}+\frac{n}{2^{2m}}\binom{2m}{m}.$$

(ii) For $x\in\mathbb{R}-\{-1,1\}$,

$$\sum_{k=1}^{n}U_{2k+1}^{2m}(x)=\frac{1}{2^{2m}(x^2-1)^m}\left[2\sum_{i=0}^{m-1}(-1)^i\binom{2m}{i}\times\frac{T_{(2m-2i)(n+3)}(x)U_{n(2m-2i)-1}(x)}{U_{2m-2i-1}(x)}+n(-1)^m\binom{2m}{m}\right].$$

Corollary 2 *For $n,m>0$,*

(i)

$$\sum_{j=1}^{n}\frac{1}{(-B)^{(2j+1)km}}v_{k(2j+1)}^{2m}=\sum_{i=0}^{m-1}\binom{2m}{i}\frac{u_{k(4m-4i)(n+1)}}{(-B)^{k(2n+1)(m-i)}u_{k(2m-2i)}}-\frac{v_k^{2m}}{(-B)^{km}}+(n+1)\binom{2m}{m}.$$

(ii)

$$\sum_{j=0}^{n}\frac{u_{2k(j+1)}^{2m}}{B^{2kmj}}=\frac{B^{2km}u_k^{2m}}{\left(v_k^2-4(-B)^k\right)^m}\left(\sum_{i=0}^{m-1}(-1)^i\binom{2m}{i}\times\frac{u_{k(2n+3)(2m-2i)}}{B^{k(2(n+1)(m-i))}u_{k(2m-2i)}}+(-1)^m\frac{2n+3}{2}\binom{2m}{m}\right).$$

If we take $A=B=k=1$ in Corollary 2, then we have the following results

(i)

$$\sum_{j=1}^{n}L_{4j+2}^{2m}=\sum_{i=0}^{m-1}\binom{2m}{i}\frac{F_{(4m-4i)(2n+2)}}{F_{4m-4i}}-3^{2m}+(n+1)\binom{2m}{m}.$$

(ii)

$$\sum_{j=0}^{n}F_{4j+4}^{2m}=\frac{1}{5^m}\left(\sum_{i=0}^{m-1}\binom{2m}{i}(-1)^i\frac{F_{(4m-4i)(2n+3)}}{F_{4m-4i}}+(n+2)(-1)^m\binom{2m}{m}-(-1)^m\binom{2m-1}{m}\right).$$

Theorem 3 *For $m,n>0$,*

(i) For even n,

$$\sum_{k=1}^{n}(-1)^k T_k^{2m}(x) = \frac{1}{2^{2m-1}}\left[(x^2-1)\sum_{i=0}^{m}\binom{2m}{i} \times \frac{U_{(n+1)(m-i)-1}(x)U_{n(m-i)-1}(x)}{T_{m-i}(x)} - \frac{(-1)^n-1}{2^2}\binom{2m}{m}\right],$$

and for odd n,

$$\sum_{k=1}^{n}(-1)^k T_k^{2m}(x) = -\frac{1}{2^{2m-1}}\left[\sum_{i=0}^{m}\binom{2m}{i} \times \frac{T_{n(m-i)}(x)T_{(n+1)(m-i)}(x)}{T_{m-i}(x)} + \frac{(-1)^n-1}{2^2}\binom{2m}{m}\right].$$

(ii) For even n and $x \in \mathbb{R}-\{-1,1\}$,

$$\sum_{k=1}^{n}(-1)^k U_k^{2m}(x) = \frac{1}{2^{2m-1}(x^2-1)^{m-1}}\sum_{i=0}^{m}(-1)^i\binom{2m}{i} \times \frac{U_{(m-i)(n+3)-1}(x)U_{n(m-i)-1}(x)}{T_{m-i}(x)}$$

and for odd n,

$$\sum_{k=1}^{n}(-1)^k U_k^{2m}(x) = \frac{1}{2^{2m}(x^2-1)^m} \times \left[(-1)^m\binom{2m}{m} - 2\sum_{i=0}^{m}(-1)^i\binom{2m}{i} \times \frac{T_{(m-i)(n+3)}(x)T_{n(m-i)}(x)}{T_{m-i}(x)}\right].$$

Proof

(i) Consider

$$\begin{aligned}
&\sum_{k=1}^{n}(-1)^k T_k^{2m}(x)\\
&=\sum_{k=1}^{n}(-1)^k\frac{1}{2^{2m}}\left[\left(x+\sqrt{x^2-1}\right)^k+\left(x-\sqrt{x^2-1}\right)^k\right]^{2m}\\
&=\sum_{k=1}^{n}\frac{(-1)^k}{2^{2m}}\sum_{i=0}^{2m}\binom{2m}{i}\left(x+\sqrt{x^2-1}\right)^{ki}\left(x-\sqrt{x^2-1}\right)^{k(2m-i)}\\
&=\sum_{k=1}^{n}\frac{(-1)^k}{2^{2m}}\sum_{i=0}^{m}\binom{2m}{i}\left(\left(x+\sqrt{x^2-1}\right)^{k(2m-i)}\left(x-\sqrt{x^2-1}\right)^{ki}\right.\\
&\left.+\left(x+\sqrt{x^2-1}\right)^{ki}\left(x-\sqrt{x^2-1}\right)^{k(2m-i)}\right)\\
&=\sum_{k=1}^{n}\frac{(-1)^k}{2^{2m}}\\
&\times\sum_{i=0}^{m}\binom{2m}{i}\left[\left(x+\sqrt{x^2-1}\right)^{2k(m-i)}+\left(x-\sqrt{x^2-1}\right)^{2k(m-i)}\right]\\
&-\frac{1}{2^{2m}}\binom{2m}{m}\sum_{k=1}^{n}(-1)^k
\end{aligned}$$

By the Binet formula of $T_n(x)$, we have

$$\begin{aligned}
&\sum_{k=1}^{n}(-1)^k T_k^{2m}(x)\\
&=\sum_{k=1}^{n}\frac{(-1)^k}{2^{2m-1}}\sum_{i=0}^{m}\binom{2m}{i}T_{2k(m-i)}(x)\\
&-\frac{1}{2^{2m}}\binom{2m}{m}\frac{(-1)^n-1}{2}\\
&=\frac{1}{2^{2m-1}}\sum_{k=1}^{m}\binom{2m}{i}\sum_{i=0}^{n}(-1)^k T_{2k(m-i)}(x)\\
&-\frac{1}{2^{2m}}\binom{2m}{m}\frac{(-1)^n-1}{2}.
\end{aligned}$$

Taking $a=2(m-i)$ and $b=0$ in Lemma 1, we write

$$\begin{aligned}
&\sum_{k=1}^{n}(-1)^k T_k^{2m}(x)\\
&=-\frac{1}{2^{2m}}\binom{2m}{m}\frac{(-1)^n-1}{2}\\
&+\frac{1}{2^{2m-1}}\sum_{i=1}^{m}\binom{2m}{i}\\
&\times\frac{T_{2(m-i)}(x)+1+(-1)^{n+1}T_{2(m-i)(n+1)}(x)}{1+T_{2(m-i)}(x)}.
\end{aligned}$$

For odd n, using (i) and (v) in Lemma 2, we have the claimed result. Similarly, for even n, the desired result is given. Now we will prove the claim (ii). Consider

$$\begin{aligned}
&\sum_{k=1}^{n}(-1)^k U_k^{2m}(x)=\sum_{k=1}^{n}(-1)^k\\
&\times\left[\frac{1}{2\sqrt{x^2-1}}\left(\left(x+\sqrt{x^2-1}\right)^{k+1}-\left(x-\sqrt{x^2-1}\right)^{k+1}\right)\right]^{2m}\\
&=\frac{1}{2^{2m}(x^2-1)^m}\sum_{k=1}^{n}(-1)^k\sum_{i=0}^{2m}\binom{2m}{i}(-1)^i\\
&\times\left(x+\sqrt{x^2-1}\right)^{(k+1)i}\left(x-\sqrt{x^2-1}\right)^{(k+1)(2m-i)}\\
&=\frac{1}{2^{2m}(x^2-1)^m}\\
&\times\left\{\sum_{k=1}^{n}(-1)^k\sum_{i=0}^{m}\binom{2m}{i}(-1)^i\right.\\
&\times\left(\left(x+\sqrt{x^2-1}\right)^{(k+1)(2m-i)}\left(x-\sqrt{x^2-1}\right)^{(k+1)i}\right.\\
&\left.\left.+\left(x+\sqrt{x^2-1}\right)^{(k+1)i}\left(x-\sqrt{x^2-1}\right)^{(k+1)(2m-i)}\right)\right.\\
&\left.-(-1)^m\binom{2m}{m}\sum_{k=1}^{n}(-1)^k\right\}\\
&=\frac{1}{2^{2m}(x^2-1)^m}\\
&\times\left[2\sum_{i=0}^{m}\binom{2m}{i}(-1)^i\sum_{k=1}^{n}(-1)^k T_{2(k+1)(m-i)}(x)-(-1)^m\right.\\
&\left.\times\binom{2m}{m}\frac{(-1)^n-1}{2}\right].
\end{aligned}$$

Taking $a = b = 2(m-i)$ in Lemma 1, we write

$$\sum_{k=1}^{n}(-1)^k U_k^{2m}(x) = -\frac{1}{4^m(x^2-1)^m}\left\{(-1)^m\binom{2m}{m}\frac{(-1)^n-1}{2}+\sum_{i=1}^{m}\binom{2m}{i}(-1)^i \times\frac{T_{4(m-i)}(x)+T_{2(m-i)}(x)+(-1)^{n+1}\left(T_{(n+2)(2m-2i)}(x)+T_{(n+1)(2m-2i)}(x)\right)}{1+T_{2(m-i)}(x)}\right\}.$$

Using (i) and (iv) in Lemma 2, we have

$$\sum_{k=1}^{n}(-1)^k U_k^{2m}(x) = -\frac{1}{4^m(x^2-1)^m} \times\left[(-1)^m\binom{2m}{m}\frac{(-1)^n-1}{2}+\sum_{i=1}^{m}\binom{2m}{i}(-1)^i \times\frac{T_{3(m-i)}(x)+(-1)^{n+1}T_{(2n+3)(m-i)}(x)}{T_{m-i}(x)}\right].$$

For even n, from (ii) and (iv) in Lemma 2, the claim is obtained. Similarly, the desired result for odd n is proven.

□

Corollary 3 *For $m, n > 0$,*

(i) For even n,

$$\sum_{j=1}^{n}(-1)^j\frac{v_{kj}^{2m}}{(-B)^{jkm}} = \sum_{i=0}^{m}\binom{2m}{i}\frac{v_{k(m-i)(2n+1)}}{(-B)^{k(m-i)n}v_{k(m-i)}} - 2^{2m-1} - \frac{2(-1)^n-1}{2}\binom{2m}{m}$$

and for odd n,

$$\sum_{j=1}^{n}(-1)^j\frac{v_{kj}^{2m}}{(-B)^{jkm}} = -\frac{1}{2^{2m}}\left(\sum_{i=0}^{m}\binom{2m}{i}\frac{v_{k(m-i)(2n+1)}}{(-B)^{k(m-i)n}v_{k(m-i)}} + 2^{2m-1} + \frac{(-1)^n}{2}\binom{2m}{m}\right).$$

(ii)

$$\sum_{j=0}^{n}(-1)^j\frac{u_{k(j+1)}^{2m}}{(-B)^{jkm}} = \frac{u_k^{2m}(-1)^n}{\left(v_k^2-4(-B)^k\right)^m}\left(\sum_{i=0}^{m}(-1)^i\binom{2m}{i} \times\frac{v_{k(2n+3)(m-i)}}{(-B)^{k(n(m-i)-i)}v_{k(m-i)}} - (-B)^{km}\frac{(-1)^m}{2}\binom{2m}{m}\right).$$

Theorem 4 *For $m, n > 0$,*

(i) For even n,

$$\sum_{k=1}^{n}(-1)^k T_{2k}^{2m}(x) = \frac{1-(-1)^n}{2^{2m+1}}\binom{2m}{m} + \frac{(x^2-1)}{2^{2m-1}}\sum_{i=0}^{m}\binom{2m}{i} \times\frac{U_{2(n+1)(m-i)-1}(x)U_{2n(m-i)-1}(x)}{T_{2(m-i)}(x)}$$

and for odd n,

$$\sum_{k=1}^{n}(-1)^k T_{2k}^{2m}(x) = \frac{1-(-1)^n}{2^{2m+1}}\binom{2m}{m} - \frac{1}{2^{2m-1}}\sum_{i=0}^{m}\binom{2m}{i}\frac{T_{2n(m-i)}(x)T_{2(n+1)(m-i)}(x)}{T_{2(m-i)}(x)}.$$

(ii) For even n,

$$\sum_{k=1}^{n}(-1)^k T_{2k}^{2m+1}(x) = \frac{(x^2-1)}{2^{2m}}\sum_{i=0}^{m}\binom{2m+1}{i} \times\frac{U_{(2m-2i+1)(n+1)-1}(x)U_{(2m-2i+1)n-1}(x)}{T_{2m-2i+1}(x)}$$

and for odd n,

$$\sum_{k=1}^{n}(-1)^k T_{2k}^{2m+1}(x) = -\frac{1}{2^{2m}} \times\sum_{i=0}^{m}\binom{2m+1}{i}\frac{T_{(2m-2i+1)(n+1)}(x)T_{(2m-2i+1)n}(x)}{T_{2m-2i+1}(x)}.$$

Corollary 4 *For $m, n > 0$,*

(i)

$$\sum_{j=1}^{n}(-1)^j\frac{1}{B^{2kmj}}v_{2kj}^{2m} = (-1)^n\sum_{i=0}^{m}\binom{2m}{i}\frac{v_{k(2m-2i)(2n+1)}}{B^{k(2m-2i)n}v_{k(2m-2i)}} - 2^{2m-1} - \frac{(-1)^n}{2}\binom{2m}{m}.$$

(ii)

$$\sum_{j=1}^{n}(-1)^j\frac{1}{(-B)^{k(2m+1)j}}v_{2kj}^{2m+1} = (-1)^n\sum_{i=0}^{m}\binom{2m+1}{i} \times\frac{v_{k(2m-2i+1)(2n+1)}}{(-B)^{k(2m-2i+1)n}v_{k(2m-2i+1)}} - 2^{2m}.$$

Theorem 5 *For $m, n > 0$,*

(i) For even n,

$$\sum_{k=1}^{n}(-1)^k T_{2k+1}^{2m}(x) = \frac{1-(-1)^n}{2^{2m+1}}\binom{2m}{m} + \frac{(x^2-1)}{2^{2m-1}} \times \sum_{i=0}^{m}\binom{2m}{i}\frac{U_{(2m-2i)n-1}(x)U_{(2m-2i)(n+2)-1}(x)}{T_{2m-2i}(x)}$$

and for odd n,

$$\sum_{k=1}^{n}(-1)^k T_{2k+1}^{2m}(x) = \frac{1-(-1)^n}{2^{2m+1}}\binom{2m}{m} - \frac{1}{2^{2m-1}}\sum_{i=0}^{m}\binom{2m}{i}\frac{T_{(2m-2i)n}(x)T_{(2m-2i)(n+2)}(x)}{T_{2m-2i}(x)}.$$

(ii) For even n,

$$\sum_{k=1}^{n}(-1)^k T_{2k+1}^{2m+1}(x) = \frac{1}{2^{2m}}(x^2-1) \times \sum_{i=0}^{m}\binom{2m+1}{i} \times \frac{U_{(2m-2i+1)n-1}(x)U_{(2m-2i+1)(n+2)-1}(x)}{T_{2m-2i+1}(x)}$$

and for odd n,

$$\sum_{k=1}^{n}(-1)^k T_{2k+1}^{2m+1}(x) = -\frac{1}{2^{2m}} \times \sum_{i=0}^{m}\binom{2m+1}{i}\frac{T_{(2m-2i+1)n}(x)T_{(2m-2i+1)(n+2)}(x)}{T_{2m-2i+1}(x)}.$$

Corollary 5 *For $m, n > 0$,*

(i)

$$\sum_{j=1}^{n}(-1)^j \frac{1}{(-B)^{k(2j+1)m}} v_{k(2j+1)}^{2m} = -\binom{2m}{m}\frac{(-1)^n-1}{2} + (-1)^n \sum_{i=0}^{m}\binom{2m}{i}(-B)^{k(i-m)} \times \frac{B^{k(2ni-2mn)}v_{k(4m-4i)(n+1)} - (-1)^n v_{k(4m-4i)}}{v_{k(2m-2i)}}.$$

(ii)

$$\sum_{j=1}^{n}(-1)^j \frac{1}{(-B)^{kj(2m+1)}} v_{k(2j+1)}^{2m+1} = (-1)^n \sum_{i=0}^{m}\binom{2m+1}{i}(-B)^{ki} \times \frac{(-B)^{kn(2i-2m-1)}v_{k(4m-4i+2)(n+1)} - v_{k(4m-4i+2)}}{v_{k(2m-2i+1)}}.$$

Theorem 6 *For $m, n > 0$ and $x \in \mathbb{R} - \{-1, 1\}$,*

(i) For even n

$$\sum_{k=1}^{n}(-1)^k U_{2k}^{2m}(x) = \frac{1}{2^{2m-1}(x^2-1)^{m-1}}\sum_{i=0}^{m}(-1)^i\binom{2m}{i} \frac{U_{2(m-i)(n+2)-1}(x)U_{2(m-i)n-1}(x)}{T_{2(m-i)}(x)}$$

and for odd n,

$$\sum_{k=1}^{n}(-1)^k U_{2k}^{2m}(x) = \frac{1}{2^{2m}(x^2-1)^m} \times \left[(-1)^m\binom{2m}{m} - 2\sum_{i=0}^{m}(-1)^i\binom{2m}{i} \times \frac{T_{2(m-i)(n+2)}(x)T_{2n(m-i)}(x)}{T_{2(m-i)}(x)}\right].$$

(ii) For even n,

$$\sum_{k=1}^{n}(-1)^k U_{2k}^{2m+1}(x) = \frac{1}{2^{2m}(x^2-1)^m}\sum_{i=0}^{m}(-1)^i\binom{2m+1}{i} \times \frac{T_{(n+2)(2m-2i+1)}(x)U_{n(2m-2i+1)-1}(x)}{T_{2m-2i+1}(x)}$$

and for odd n,

$$\sum_{k=1}^{n}(-1)^k U_{2k}^{2m+1}(x) = \frac{-1}{2^{2m}(x^2-1)^m}\sum_{i=0}^{m}(-1)^i\binom{2m+1}{i} \times \frac{U_{(n+2)(2m-2i+1)-1}(x)T_{n(2m-2i+1)}(x)}{T_{2m-2i+1}(x)}.$$

Corollary 6 *For* $m, n > 0$,

(i) For even n,

$$\sum_{j=1}^{n}(-1)^j\frac{u_{k(2j+1)}^{2m}}{B^{2kmj}} = \frac{u_k^{2m}}{\left(v_k^2-4(-B)^k\right)^m} \times \sum_{i=0}^{m}(-1)^i\binom{2m}{i}\frac{v_{4k(n+1)(m-i)}-B^{2kn(m-i)}v_{4k(m-i)}}{(-B)^{k(2n(m-i)-i)}v_{2k(m-i)}},$$

and for odd n,

$$\sum_{j=1}^{n}(-1)^j\frac{u_{k(2j+1)}^{2m}}{B^{2kmj}} = \frac{u_k^{2m}}{\left(v_k^2-4(-B)^k\right)^m} \times \left((-1)^m(-B)^{km}\binom{2m}{m} - \sum_{i=0}^{m}(-1)^i\binom{2m}{i} \times \frac{v_{4k(n+1)(m-i)}+B^{2kn(m-i)}v_{4k(m-i)}}{(-B)^{k(2n(m-i)-i)}v_{2k(m-i)}}\right)$$

(ii)

$$\sum_{j=0}^{n}(-1)^j\frac{u_{k(2j+1)}^{2m+1}}{(-B)^{kj(2m+1)}} = \frac{u_k^{2m}}{\left(v_k^2-4(-B)^k\right)^m} \times \sum_{i=0}^{m}(-1)^{i+n}\binom{2m+1}{i}\frac{u_{2k(n+1)(2m-2i+1)}}{(-B)^{k(n(2m-2i+1)-i)}v_{k(2m-2i+1)}}.$$

If we take $A = B = k = 1$ in Corollary 6, then, for odd n, we obtain

$$\sum_{k=1}^{n}(-1)^k F_{4k+2}^{2m} = 1 - 5^{1-m}\sum_{i=0}^{m}(-1)^i\binom{2m}{i}\frac{F_{(4m-4i)(n+1)}^2}{L_{4m-4i}}.$$

Theorem 7 *For* $m, n > 0$ *and* $x \in \mathbb{R} - \{-1, 1\}$,

(i) For even n,

$$\sum_{k=1}^{n}(-1)^k U_{2k+1}^{2m}(x) = \frac{1}{2^{2m-1}(x^2-1)^{m-1}} \times \sum_{i=0}^{m}(-1)^i\binom{2m}{i}\frac{U_{2(m-i)(n+3)-1}(x)U_{2n(m-i)-1}(x)}{T_{2(m-i)}(x)},$$

and for odd n,

$$\sum_{k=1}^{n}(-1)^k U_{2k+1}^{2m}(x) = \frac{1}{2^{2m}(x^2-1)^m} \times \left[(-1)^m\binom{2m}{m} - \sum_{i=0}^{m}(-1)^i\binom{2m}{i} \times \frac{2T_{2(m-i)(n+3)}(x)T_{2n(m-i)}(x)}{T_{2(m-i)}(x)}\right].$$

(ii) For even n,

$$\sum_{k=1}^{n}(-1)^k U_{2k+1}^{2m+1}(x) = \frac{1}{2^{2m}(x^2-1)^m} \times \sum_{i=0}^{m}(-1)^i\binom{2m+1}{i} \times \frac{T_{(n+3)(2m-2i+1)}(x)U_{n(2m-2i+1)-1}(x)}{T_{2m-2i+1}(x)}.$$

and for odd n,

$$\sum_{k=1}^{n}(-1)^k U_{2k+1}^{2m+1}(x) = \frac{-1}{2^{2m}(x^2-1)^m} \times \sum_{i=0}^{m}(-1)^i\binom{2m+1}{i}\frac{U_{(n+3)(2m-2i+1)-1}(x)T_{n(2m-2i+1)}(x)}{T_{2m-2i+1}(x)}.$$

Corollary 7 *For* $m, n > 0$,

(i)

$$\sum_{j=0}^{n}(-1)^j\frac{u_{2k(j+1)}^{2m}}{B^{2kmj}}=\frac{u_k^{2m}(-1)^n}{\left(v_k^2-4(-B)^k\right)^m}\times\left(\sum_{i=0}^{m}(-1)^i\binom{2m}{i}\frac{v_{2k(2n+3)(m-i)}}{B^{k(2n(m-i)-2i)}v_{2k(m-i)}}-\frac{1}{2}(-1)^m\binom{2m}{m}B^{2km}\right).$$

(ii)

$$\sum_{j=1}^{n}(-1)^j\frac{u_{2k(j+1)}^{2m+1}}{(-B)^{kj(2m+1)}}=\frac{u_k^{2m}}{\left(v_k^2-4(-B)^k\right)^m}\sum_{i=0}^{m}(-1)^i\binom{2m+1}{i}\times\left((-1)^n\frac{u_{k(2n+3)(2m-2i+1)}}{(-B)^{k(n(2m-2i+1)-2i)}v_{k(2m-2i+1)}}\times-B^{2ki}\frac{u_{3k(2m-2i+1)}}{v_{k(2m-2i+1)}}\right).$$

References

Kılıç E, Stănică P (2009) Factorizations and representations of second order linear recurrences with indices in arithmetic progressions. Bull Mexican Math Soc 15(1):23–36

Kılıç E, Stănică P (2011) Factorizations and representations of binary polynomial recurrences by matrix methods. Rocky Mountain J Math 41(4):1247–1264

Kılıç E, Ömür N, Ulutaş YT (2011) Alternating sums of the powers of Fibonacci and Lucas numbers. Miskolc Math Notes 12(1):87–103

Li X (2015) Some identities involving Chebyshev polynomials. Math Probl Eng Article ID 950695

Ma R, Zhang W (2007) Several identities involving the Fibonacci numbers and Lucas numbers. Fibonacci Q 45:164–170

Prodinger H (2008) On a sum of Melham and its variants. Fibonacci Q 46/47(3):207–215

Wang T, Zhang H (2015) Some identities involving the derivative of the first kind Chebyshev polynomials. Math Probl Eng, Article ID 146313

Zhang W, Wang T (2017) Two identities involving the integral of the first kind Chebyshev polynomials. Bull Math Sci Math Roumanie 108(1):91–98