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ON CONGRUENCES INVOLVING THE GENERALIZED CATALAN NUMBERS AND HARMONIC NUMBERS

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Abstract. In this paper, we prove some congruences involving the generalized Catalan numbers and harmonic numbers modulo p^2, one of which is

$$\sum_{k=1}^{p-1} k^2 B_{p,k} B_{p,k-d} \equiv 4\,(-1)^d \left\{ \frac{1}{3} d\left(2d^2+1\right)(4pH_d - 1) - p\left(\frac{26}{9}d^3 + \frac{4}{3}d^2 + \frac{7}{9}d + \frac{1}{2}\right)\right\} \pmod{p^2},$$

where a prime number $p > 3$ and $1 \le d \le p$.

1. Introduction

The Catalan numbers have important applications in combinatorics, number theory and the analysis algorithms in [5, 6, 8, 10].

In [10], Shapiro gave the generalized Catalan numbers $B_{n,k}$ as follows:

$$B_{n,k} = \frac{k}{n}\binom{2n}{n-k},\ 0 \le k \le n.$$

The numbers $B_{n,k}$ are the entries of the Catalan triangles and satisfy the recurrence relation

$$B_{n,k} = B_{n-1,k-1} + 2B_{n-1,k} + B_{n-1,k+1}, \quad k \ge 2$$

with the initial conditions $B_{n,0} = 0 = B_{n,n+m}$, $m \ge 1$. The numbers $B_{n,k}$ are not known as Catalan numbers. However they have several applications [3, 7, 10].

Note that for $k = 1$, $B_{n,1}$ are the well known Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \ge 1.$$

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Harmonic numbers are those rational numbers given by

$$H_0 = 0, \quad H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{Z}^+,$$

and for $m \in \mathbb{Z}^+$, harmonic numbers of order m are those rational numbers

$$H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}, \quad n \in \mathbb{Z}^+.$$

Some interesting properties of harmonic numbers are given as follows [1,2,4,11]:

$$\sum_{k=1}^{n-1} \frac{k^{\underline{m}}}{n-k} H_k = n^{\underline{m}} \left(H_n \left(H_n - H_m \right) - H_{n,2} + H_{m,2} \right), \tag{1}$$

$$\sum_{k=1}^{n-1} k^{\underline{m}} H_{n-k} = n^{\underline{m}} \frac{n+1}{m+1} \left(H_{n+1} - H_{m+1} \right), \tag{2}$$

$$\sum_{k=1}^{n} k^{\underline{m}} H_k = n^{\underline{m}} \frac{n+1}{m+1} \left(H_{n+1} - \frac{1}{m+1} \right), \tag{3}$$

$$\sum_{k=1}^{n} \frac{H_k}{k} = \frac{1}{2} \left(H_n^2 + H_{n,2} \right), \tag{4}$$

where $x^{\underline{m}}$ stands for the falling factorial defined by

$$x^{\underline{m}} = x \left(x-1 \right) \cdots \left(x-m+1 \right).$$

In [12], Sun showed that for any odd prime p and $0 \leq k < p$,

$$(-1)^k \binom{p-1}{k} = \prod_{0<j\leq k} \left(1 - \frac{p}{j} \right) \equiv 1 - pH_k \pmod{p^2}. \tag{5}$$

In [6], Gutiérrez et al. gave some identities involving well-known Catalan numbers. For example,

$$\sum_{i=1}^{m} B_{n,i} B_{n,n+i-m} \left(n + 2i - m \right) = \left(n+1 \right) C_n \binom{2(n-1)}{m-1}, \ m \leq n,$$

$$\sum_{i=1}^{n} i \left(B_{n,i} \right)^2 = \frac{n \left(n+1 \right)}{2} C_n C_{n-1}.$$

In [8], Miana and Romero showed the following identity that for $1 \leq m \leq n$,

$$\sum_{i=1}^{m} B_{n,i} B_{n,n+i-m} \left(n + 2i - m \right)^3 = \binom{2n}{n} \binom{2(n-1)}{m-1} \left(n^2 + 4n - 2nm + m^2 \right).$$

In [9], Ömür and Koparal obtained various congruences involving the numbers $B_{p,k-d}$. For example, for $1 \leq d \leq p-1$,

$$\sum_{k=1}^{p-1} B_{p,k}B_{p,k-d} \equiv 4\,(-1)^d \left(-2d + p\left(8d\,(H_d - 1) + \frac{2}{d} - 3\right)\right) \pmod{p^2}$$

and for $1 \leq d \leq p$,

$$\sum_{k=1}^{p-1} kB_{p,k}B_{p,k-d} \equiv 2\,(-1)^d \left(-2d^2 + p\left(1 + 8d^2\,(H_d - 1) - 4d\right)\right) \pmod{p^2}.$$

In this paper, we prove some congruences involving the generalized Catalan numbers and harmonic numbers modulo p^2, one of which is

$$\sum_{k=1}^{p-1} k^2 B_{p,k}B_{p,k-d} \equiv 4\,(-1)^d \left\{\frac{1}{3}d\left(2d^2+1\right)(4pH_d - 1) - p\left(\frac{26}{9}d^3 + \frac{4}{3}d^2 + \frac{7}{9}d + \frac{1}{2}\right)\right\} \pmod{p^2},$$

where a prime number $p > 3$ and $1 \leq d \leq p$.

2. Some congruences involving the generalized Catalan numbers and harmonic numbers

In this section, firstly, we will continue our work with the following lemmas.

Lemma 2.1. *Let $p > 3$ be a prime number. For $0 \leq m < p-1$ and $1 \leq d \leq p-1$, we have*

$$\sum_{k=d}^{p-1} k^{\underline{m}} H_{k-d} \equiv (p+d-1)^{\underline{m}} \frac{p+d}{p\,(m+1)} - (d-1)^{\underline{m}} \frac{d}{m+1}\left(\frac{1}{p} + H_d - H_{m+1}\right) \pmod{p}.$$

Proof. Observed that

$$\sum_{k=1}^{p+d-2} k^{\underline{m}} H_{p+d-1-k} = \sum_{k=1}^{d-1} k^{\underline{m}} H_{p+d-1-k} + \sum_{k=d}^{p-1} k^{\underline{m}} H_{p+d-1-k} + \sum_{k=p}^{p+d-2} k^{\underline{m}} H_{p+d-1-k}.$$

From the congruences for $1 \leq k,\ m < p$,

$$H_{p-k} \equiv H_{k-1} \pmod{p},\ (p+k)^{\underline{m}} \equiv k^{\underline{m}} \pmod{p} \text{ and } H_{p+k} \equiv H_k + \frac{1}{p} \pmod{p}, \tag{6}$$

we have

$$\sum_{k=1}^{p+d-2} k^{\underline{m}} H_{p+d-1-k} \equiv \sum_{k=1}^{d-1} k^{\underline{m}} \left(H_{d-1-k} + \frac{1}{p} \right) + \sum_{k=d}^{p-1} k^{\underline{m}} H_{k-d} + \sum_{k=0}^{d-2} k^{\underline{m}} H_{d-1-k}$$
$$= 2 \sum_{k=1}^{d-2} k^{\underline{m}} H_{d-1-k} + \frac{1}{p} \sum_{k=1}^{d-1} k^{\underline{m}} + \sum_{k=d}^{p-1} k^{\underline{m}} H_{k-d} \pmod{p}.$$

Considering (2) and (6), we write

$$\sum_{k=d}^{p-1} k^{\underline{m}} H_{k-d} \equiv \sum_{k=1}^{p+d-2} k^{\underline{m}} H_{p+d-1-k} - 2 \sum_{k=1}^{d-2} k^{\underline{m}} H_{d-1-k} - \frac{1}{p} \sum_{k=1}^{d-1} k^{\underline{m}}$$
$$\equiv -(d-1)^{\underline{m}} \frac{d}{m+1} \left(H_d - H_{m+1} \right) + (p+d-1)^{\underline{m}} \frac{p+d}{p\,(m+1)}$$
$$- \frac{1}{p} \sum_{k=1}^{d-1} k^{\underline{m}} \pmod{p}.$$

It follows from the sum $\sum_{k=a}^{n} k^{\underline{m}} = \frac{1}{m+1} \left((n+1) n^{\underline{m}} - a (a-1)^{\underline{m}} \right)$ that

$$\sum_{k=d}^{p-1} k^{\underline{m}} H_{k-d}$$
$$\equiv (p+d-1)^{\underline{m}} \frac{p+d}{p\,(m+1)} - (d-1)^{\underline{m}} \frac{d}{m+1} \left(\frac{1}{p} + H_d - H_{m+1} \right) \pmod{p}.$$

Thus, we complete the proof. □

Lemma 2.2. *Let p be an odd prime. For $1 < d \leq p-1$, we have*

$$\sum_{k=d}^{p-1} \frac{1}{k} H_{k-d} \equiv H_{p+d-1}^2 - H_{p+d-1,2} - 2H_{d-1}^2 + 2H_{d-1,2} - \frac{2}{p} H_{d-1} \pmod{p}.$$

Proof. We write

$$\sum_{k=1}^{p+d-2} \frac{1}{k} H_{p+d-k-1} = \sum_{k=1}^{d-1} \frac{1}{k} H_{p+d-k-1} + \sum_{k=d}^{p-1} \frac{1}{k} H_{p+d-k-1} + \sum_{k=p}^{p+d-2} \frac{1}{k} H_{p+d-k-1}.$$

From the congruences $H_{p+k} \equiv H_k + \frac{1}{p} \pmod{p}$ and $H_{p-k} \equiv H_{k-1} \pmod{p}$ for $1 \leq k < p$, we get

$$\sum_{k=1}^{p+d-2} \frac{1}{k} H_{p+d-k-1} \equiv \sum_{k=1}^{d-1} \frac{1}{k} \left(H_{d-k-1} + \frac{1}{p} \right) + \sum_{k=d+1}^{p-1} \frac{1}{k} H_{k-d}$$
$$+ \frac{H_{d-1}}{p} + \sum_{k=1}^{d-2} \frac{1}{p+k} H_{d-k-1} \pmod{p}.$$

By the congruence $\frac{1}{p+k} \equiv \frac{1}{k} \pmod{p}$ for $1 \leq k < p$, we have

$$\sum_{k=1}^{p+d-2} \frac{1}{k} H_{p+d-k-1} \equiv 2 \sum_{k=1}^{d-2} \frac{1}{k} H_{d-k-1} + \sum_{k=d+1}^{p-1} \frac{1}{k} H_{k-d} + 2\frac{H_{d-1}}{p} \pmod{p}.$$

Considering the sum

$$\sum_{k=1}^{n-1} \frac{1}{k} H_{n-k} = H_n^2 - H_{n,2} \tag{7}$$

from (1), we get

$$H_{p+d-1}^2 - H_{p+d-1,2} \equiv 2\left(H_{d-1}^2 - H_{d-1,2}\right) + \sum_{k=d+1}^{p-1} \frac{1}{k} H_{k-d} + 2\frac{H_{d-1}}{p} \pmod{p}.$$

Thus the desired result is obtained. □

Lemma 2.3. *For $n \in \mathbb{N}$, we have*

$$\sum_{k=1}^{n} (-1)^k k^2 H_k = (-1)^n \binom{n+1}{2} H_n + (-1)^n \frac{n}{4} + \frac{1}{8}\left(1 - (-1)^n\right).$$

Proof. Observed that

$$\begin{aligned} \sum_{k=1}^{n} (-1)^k k^2 H_k &= \sum_{k=1}^{n} (-1)^k k^2 \sum_{j=1}^{k} \frac{1}{j} = \sum_{j=1}^{n} \frac{1}{j} \sum_{k=j}^{n} (-1)^k k^2 \\ &= \sum_{j=1}^{n} \frac{1}{j} \sum_{k=1}^{n} (-1)^k k^2 - \sum_{j=1}^{n} \frac{1}{j} \sum_{k=1}^{j-1} (-1)^k k^2. \end{aligned}$$

Applying $\sum_{k=1}^{n} (-1)^k k^2 = (-1)^n \binom{n+1}{2}$, we write

$$\sum_{k=1}^{n} (-1)^k k^2 H_k = (-1)^n \binom{n+1}{2} H_n + \frac{1}{2} \sum_{j=1}^{n} (-1)^j (j-1).$$

By the sums

$$\sum_{k=1}^{n} (-1)^k k = -\frac{1}{4} + \frac{(-1)^n}{2}\left(n + \frac{1}{2}\right) \quad \text{and} \quad \sum_{k=1}^{n} (-1)^k = \frac{(-1)^n - 1}{2},$$

we conclude the proof. □

Now we give some congruences involving harmonic numbers in the following theorems.

Theorem 2.4. *Let p be an odd prime. For $1 < d < p$, we have*

$$\sum_{k=1}^{p-1} \frac{B_{p,k} B_{p,k-d}}{k} \equiv 4(-1)^d \left\{ H_d + H_{d-1} + \frac{2}{p} + p\left(4H_d^2 + 4H_{d-1}^2\right.\right.$$

$$-2H_{d,2}-2H_{d-1,2}-H_{p+d}^2-H_{p+d-1}^2\big)\big\} \pmod{p^2},$$

and for $3<d\leq p$,

$$\begin{aligned}&\sum_{k=1}^{p-1}(-1)^k k^2 B_{p,k}B_{p,k-d}\\ \equiv\ & -4d+4p\left\{1+\frac{(-1)^d}{2}+\binom{d+1}{2}\left(H_{\lfloor\frac{d-1}{2}\rfloor}+H_{\lfloor\frac{p+d-1}{2}\rfloor}\right)\right.\\ &\left.-\binom{d}{2}\left(H_{\lfloor\frac{d-2}{2}\rfloor}+H_{\lfloor\frac{p+d-2}{2}\rfloor}\right)\right\} \pmod{p^2}.\end{aligned}$$

Proof. From $B_{p,k-d}=-B_{p,d-k}$ for $k<d$, we write

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{B_{p,k}B_{p,k-d}}{k} &= \sum_{k=1}^{d}\frac{B_{p,k}B_{p,k-d}}{k}+\sum_{k=d+1}^{p-1}\frac{B_{p,k}B_{p,k-d}}{k}\\ &= -\sum_{k=1}^{d-2}\frac{B_{p,k}B_{p,d-k}}{k}+\sum_{k=d+1}^{p-1}\frac{B_{p,k}B_{p,k-d}}{k}\\ &\quad -\frac{1}{p^2}\binom{2p}{p-1}\binom{2p}{p-d+1}.\end{aligned}$$

For $1<k\leq p$ and $d+2\leq k$, using the congruence

$$B_{p,k-d}\equiv 2\,(-1)^{d-k-1}\,(1-p\,(H_{k-d}+H_{k-1-d})) \pmod{p^2}, \tag{8}$$

in [9], we have

$$\begin{aligned}&\sum_{k=1}^{p-1}\frac{B_{p,k}B_{p,k-d}}{k}\\ &\equiv 4\,(-1)^d\left\{p\sum_{k=1}^{d-2}\frac{H_k+H_{k-1}}{k}+p\sum_{k=1}^{d-2}\frac{H_{d-k}+H_{d-k-1}}{k}\right.\\ &\quad\left.+\sum_{k=d+1}^{p-1}\frac{1}{k}-\sum_{k=1}^{d-2}\frac{1}{k}-p\sum_{k=d+1}^{p-1}\frac{H_{k-d}+H_{k-d-1}}{k}-p\sum_{k=d+1}^{p-1}\frac{H_k+H_{k-1}}{k}\right\}\\ &\quad-\frac{1}{p^2}\binom{2p}{p-1}\binom{2p}{p-d+1} \pmod{p^2}.\end{aligned}$$

Using (4), (7) and Lemma 2.2, we write

$$\begin{aligned}&\sum_{k=1}^{p-1}\frac{B_{p,k}B_{p,k-d}}{k}\\ &\equiv 4\,(-1)^d\left\{pH_{d-2}^2+p\left(H_d^2+H_{d-1}^2-H_{d,2}-H_{d-1,2}-\frac{1}{d-1}\right)+H_{p-1}-H_d-H_{d-2}\right.\\ &\quad\left.-p\left(H_{p+d-1}^2+H_{p+d}^2-H_{p+d-1,2}-H_{p+d,2}-2\left(H_{d-1}^2+H_d^2-H_{d-1,2}-H_{d,2}\right)\right)\right.\end{aligned}$$

$$+2(H_{d-1}+H_d)-p(H_{p-1}^2-H_d^2)\}-\frac{1}{p^2}\binom{2p}{p-1}\binom{2p}{p-d+1} \pmod{p^2}.$$

Since the congruences $H_{p+k,2}\equiv H_{k,2}+\frac{1}{p^2} \pmod p$ for $1\le k\le p-1$ and $H_{p-1}\equiv 0 \pmod{p^2}$, we have

$$\begin{aligned}
&\sum_{k=1}^{p-1}\frac{B_{p,k}B_{p,k-d}}{k}\\
&\equiv 4(-1)^d\left\{\frac{2}{p}-H_{d-2}+2H_{d-1}+H_d+p\left(H_{d-2}^2+4H_d^2-\frac{1}{d-1}\right)\right.\\
&\qquad\left.-p\left(H_{p+d-1}^2-3H_{d-1}^2+2H_{d-1,2}+H_{p+d}^2+2H_{d,2}\right)\right\}\\
&\qquad-\frac{1}{p^2}\binom{2p}{p-1}\binom{2p}{p-d+1} \pmod{p^2}.
\end{aligned}$$

For $1<d\le p$, taking the congruence

$$\begin{aligned}
&\frac{d-1}{p^2}\binom{2p}{p-d+1}\binom{2p}{p-1}\\
&\equiv 4(-1)^d\left(1+\frac{p}{d-1}+\frac{2p}{d}-p-2pH_d\right) \pmod{p^2},
\end{aligned}\tag{9}$$

in [9], we obtain the desired result. With the help of Lemma 2.3, the proof of the other result is given. □

For example, for $p\ge 5$ and $d=4$ in Theorem 2.4, we obtain the congruences

$$\sum_{k=1}^{p-1}\frac{B_{p,k}B_{p,k-4}}{k}\equiv 4\left(\frac{47}{12}+\frac{2}{p}+p\left(\frac{1817}{72}-H_{p+4}^2-H_{p+3}^2\right)\right) \pmod{p^2}$$

and

$$\sum_{k=1}^{p-1}(-1)^k k^2 B_{p,k}B_{p,k-4}\equiv -16+4p\left\{\frac{11}{2}+10H_{\lfloor\frac{p+3}{2}\rfloor}-6H_{\lfloor\frac{p+2}{2}\rfloor}\right\} \pmod{p^2}.$$

Theorem 2.5. *Let $p>3$ be a prime number. For $0<d\le p$ and $0<m<p-1$, we have*

$$\begin{aligned}
&\sum_{k=1}^{p-1}k^{\underline{m}}B_{p,k}B_{p,k-d}\\
&\equiv\frac{4(-1)^d}{m+1}\{p(m-1)!(-1)^m-(p+d)(p+d-1)^{\underline{m}}-(p+d+1)(p+d)^{\underline{m}}\\
&\qquad+2p(2H_d-H_m)((d+1)d^{\underline{m}}+d(d-1)^{\underline{m}})\\
&\qquad-p\left(\frac{8d}{(m+1)}(d-1)^{\underline{m}}+\frac{m+1}{m}(d^{\underline{m}}+(d-1)^{\underline{m}})\right)\Big\} \pmod{p^2}.
\end{aligned}$$

Proof. Observed that

$$\sum_{k=1}^{p-1} k^{\underline{m}} B_{p,k} B_{p,k-d} = \sum_{k=1}^{d} k^{\underline{m}} B_{p,k} B_{p,k-d} + \sum_{k=d+1}^{p-1} k^{\underline{m}} B_{p,k} B_{p,k-d}.$$

From $B_{p,k-d} = -B_{p,d-k}$ for $k < d$, we get

$$\begin{aligned}\sum_{k=1}^{p-1} k^{\underline{m}} B_{p,k} B_{p,k-d} = & -\sum_{k=1}^{d-2} k^{\underline{m}} B_{p,k} B_{p,d-k} + \sum_{k=d+1}^{p-1} k^{\underline{m}} B_{p,k} B_{p,k-d} \\ & - \frac{d-1}{p^2} (d-1)^{\underline{m}} \binom{2p}{p-d+1} \binom{2p}{p-1}.\end{aligned}$$

It follows from (8) that

$$\begin{aligned}&\sum_{k=1}^{p-1} k^{\underline{m}} B_{p,k} B_{p,k-d} \\ &\equiv 4(-1)^d \left\{ -\sum_{k=1}^{d-2} k^{\underline{m}} + p \sum_{k=1}^{d-2} k^{\underline{m}} (H_k + H_{k-1}) + p \sum_{k=1}^{d-2} k^{\underline{m}} (H_{d-k} + H_{d-k-1}) \right. \\ &\qquad \left. + \sum_{k=d+1}^{p-1} k^{\underline{m}} - p \sum_{k=d+1}^{p-1} k^{\underline{m}} (H_{k-d} + H_{k-d-1}) - p \sum_{k=d+1}^{p-1} k^{\underline{m}} (H_k + H_{k-1}) \right\} \\ &\qquad - \frac{d-1}{p^2} (d-1)^{\underline{m}} \binom{2p}{p-d+1} \binom{2p}{p-1} \pmod{p^2}.\end{aligned}$$

According to (2), (3) and Lemma 2.1, using the sum

$$\sum_{k=a}^{n} k^{\underline{m}} = \frac{1}{m+1} \left((n+1) n^{\underline{m}} - a(a-1)^{\underline{m}} \right),$$

we obtain

$$\begin{aligned}&\sum_{k=1}^{p-1} k^{\underline{m}} B_{p,k} B_{p,k-d} \\ &\equiv 4(-1)^d \left\{ -\frac{(d-1)(d-2)^{\underline{m}}}{m+1} + 2p \left(\frac{(d-1)(d-2)^{\underline{m}}}{m+1} \left(H_{d-1} - \frac{1}{m+1} \right) \right) \right. \\ &\quad + p \left(\frac{2(d+1) d^{\underline{m}}}{m+1} (H_{d+1} - H_{m+1}) + \frac{2d(d-1)^{\underline{m}}}{m+1} (H_d - H_{m+1}) - (d-1)^{\underline{m}} \right) \\ &\quad - p \left(\frac{(p+d)(p+d-1)^{\underline{m}}}{p(m+1)} - \frac{d(d-1)^{\underline{m}}}{p(m+1)} + \frac{(p+d+1)(p+d)^{\underline{m}}}{p(m+1)} - \frac{(d+1) d^{\underline{m}}}{p(m+1)} \right) \\ &\quad - p \left(\frac{2p(p-1)^{\underline{m}}}{m+1} \left(H_p - \frac{1}{m+1} \right) - 2\frac{d+1}{m+1} d^{\underline{m}} \left(H_{d+1} - \frac{1}{m+1} \right) - \frac{(p-1)^{\underline{m}} - d^{\underline{m}}}{m} \right) \\ &\quad \left. + \left(\frac{1}{m+1} (p(p-1)^{\underline{m}} - (d+1) d^{\underline{m}}) \right) - \frac{p}{m} (d-2)^{\underline{m}} \right\}\end{aligned}$$

$$-\frac{d-1}{p^2}(d-1)^{\underline{m}}\binom{2p}{p-d+1}\binom{2p}{p-1} \pmod{p^2}.$$

From (5) and (9), we write

$$\begin{aligned}
&\sum_{k=1}^{p-1} k^{\underline{m}} B_{p,k}B_{p,k-d}\\
&\equiv 4(-1)^d\left\{-\frac{(d-1)(d-2)^{\underline{m}}}{m+1}+2p\left(\frac{(d-1)(d-2)^{\underline{m}}}{m+1}\left(H_{d-1}-\frac{1}{m+1}\right)\right)\right.\\
&\quad +p\left(\frac{2(d+1)d^{\underline{m}}}{m+1}(H_{d+1}-H_{m+1})+\frac{2d(d-1)^{\underline{m}}}{m+1}(H_d-H_{m+1})-(d-1)^{\underline{m}}\right)\\
&\quad -\frac{p+d}{m+1}(p+d-1)^{\underline{m}}+\frac{d(d-1)^{\underline{m}}}{m+1}-\frac{p+d+1}{m+1}(p+d)^{\underline{m}}\\
&\quad -\frac{p}{m+1}m!(-1)^m-p\left(-2\frac{d+1}{m+1}d^{\underline{m}}\left(H_{d+1}-\frac{1}{m+1}\right)-\frac{1}{m}\left((-1)^m m!-d^{\underline{m}}\right)\right)\\
&\quad \left.-(d-1)^{\underline{m}}\left(\frac{p}{d-1}+\frac{2p}{d}+1-p-2pH_d\right)-\frac{p}{m}(d-2)^{\underline{m}}\right\} \pmod{p^2}.
\end{aligned}$$

Using the identities $\binom{n}{k}=\frac{n}{k}\binom{n-1}{k-1}$ and $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$, we have

$$\begin{aligned}
&\sum_{k=1}^{p-1} k^{\underline{m}} B_{p,k}B_{p,k-d}\\
&\equiv \frac{4(-1)^d}{m+1}\{p(m-1)!(-1)^m-(p+d)(p+d-1)^{\underline{m}}-(p+d+1)(p+d)^{\underline{m}}\\
&\qquad +2p(2H_d-H_m)((d+1)d^{\underline{m}}+d(d-1)^{\underline{m}})\\
&\qquad -p\left(\frac{8d}{(m+1)}(d-1)^{\underline{m}}+\frac{m+1}{m}(d^{\underline{m}}+(d-1)^{\underline{m}})\right)\Big\} \pmod{p^2}.
\end{aligned}$$

This completes the proof. □

Corollary 2.6. *Let $p>3$ be a prime number. For $0<d\le p$, we have*

$$\begin{aligned}
\sum_{k=1}^{p-1} k^2 B_{p,k}B_{p,k-d} \equiv 4(-1)^d\Big\{&\frac{1}{3}d\left(2d^2+1\right)(4pH_d-1)\\
&-p\left(\frac{26}{9}d^3+\frac{4}{3}d^2+\frac{7}{9}d+\frac{1}{2}\right)\Big\} \pmod{p^2}.
\end{aligned}$$

Proof. When $m=2$ in Theorem 2.5, using the congruence

$$\sum_{k=1}^{p-1} kB_{p,k}B_{p,k-d} \equiv 2(-1)^d\left(-2d^2+p\left(1+8d^2(H_d-1)-4d\right)\right) \pmod{p^2}$$

in [9], the proof easily is obtained. □

For example, for $p > 3$, $d = 2$ in Corollary 2.6, we have

$$\sum_{k=1}^{p-1} k^2 B_{p,k} B_{p,k-2} \equiv 2\,(11p - 12) \pmod{p^2}.$$

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