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Predicting nonlinear phenomena through fractional derivatives and advanced Riccati mapping: bifurcation and solitary wave behavior in nonlinear systems

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Abstract

This paper introduces a new mathematical analysis for approximating and predicting nonlinear dynamical systems through fractional calculus coupled with diffusion-reaction equations. The idea is to discuss specific aspects of system behaviors, memory effects, and historical dependencies for fractional-order systems. We apply the Generalized Riccati Equation Mapping Method (GREMM) for imaging the nonlinear system and developing more insights into the complex systems' behaviors as well as memory effects, especially in the case of fractional-order systems. This methodology, thus, allows us to get a better understanding. More than that, the current approaches introduced to understand the behavior of sophisticated systems do so far better compared to the traditional methodologies to study and make a prognosis on the behavior of solitary waves in these systems. The extended GREMM is also important for the analysis as it allows for the investigation of complex wave processes, such as solitary waves and solitons. We begin by describing kink waves and pulse waves, as well as linear and nonlinear periodic waves, and how they behave. We also discuss bifurcation analysis in which a relatively small change in the value of the parameters leads to large changes in the behavior, up to chaos by introducing fractional derivatives, and the method assists in the advancement of the analysis, the prediction, and the control of many processes in science and engineering disciplines, such as biology, chemistry, and physics. It provides better approximations. As compared to other methods, this method can provide better predictions of nonlinear phenomena for controlling power interactions and any other system demonstrating nonlinear behavior.

Keywords: Nonlinear dynamical systems; Diffusion-reaction equations; Fractional derivatives; Fractional order systems

1 Introduction

Nonlinear dynamic systems, as studied by E. Infeld et al. [19], are one of the most important theories today and are of great relevance in both physical and life sciences, engineer-

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ing, and environmental science. These systems are known to possess complex attributes, including chaos [13] and bifurcation [9], as explored by R. Ghandour et al. and M. Esteban et al. in their respective studies, and hysteresis and, in general, are difficult to predict and analyze. The nonlinear partial differential equations (PDEs) discussed in [14, 29] have been extensively studied by N. Mhadhbi et al. and A. Aigner et al. Highlighting their significance in modeling complex phenomena forms the theoretical basis for such systems, representing temporal and spatial dependencies. Some traditional modeling approaches, which employ ordinary differential equations (ODEs) as discussed in [30], have been explored by H. Miao et al. Showcasing their utility in simplifying complex systems or PDEs to capture the dependencies in the systems may not be sufficient to describe the system in question, particularly when it invokes behavior in which the nonlocal or time-related history of the information content in the context of the system is an important factor. Nonlinear PDEs form the theoretical basis of systems that represent temporal and spatial dependencies. The response properties of nonlinear PDEs are qualitatively different from those of linear PDEs. The work also explains that the superposition principle holds in linear systems; thus, the total reaction to multiple input stimuli is the sum of the discrete responses. This characteristic makes such models predictable and allows for simple analytical solutions. In nonlinear PDEs, the terms interconnect in an unproportional manner, leading to such effects as the material complexity encompassing chaotic dynamics, patterning, and self-organizing capabilities. For example, the Navier-Stokes equations used to describe fluid dynamics present turbulence and other behaviors that a linear system cannot predict. These equations describe actual processes and therefore require at least complex mathematical calculation procedures.

In the physical context, the nonlinear fractional diffusion-reaction equation discussed in this paper covers various processes in which memory and nonlocality effects prevail. Such equations are also important for the description of different quantitative aspects of various physical and biological processes, such as the diffusion of anomalous diffusion in body tissues, charge transport through heterogeneous media, the propagation and the dispersion of waves in plasmas and fluids. Their use in system modeling ensures that the dependency current on the immediate past is not the only factor controlling it, but, instead, the current state depends on any prior states. The interaction of nonlinear terms is also used to describe self-interaction and collectivity features quite characteristic of chemical reactions, ecological systems, and energy transport processes. Hence, the mathematical structure described in this work is relevant in physics, biology, and different fields of engineering.

Some of the striking solutions associated with nonlinear PDEs are solitons, known for their stability [18, 33], as investigated by M. Inc et al. and Z. Ur Rehman et al. in their studies of compact forms, which retain their forms while in motion. Solitons are obtained from a delicate interplay between nonlinearities, which work to build up the wave, and dispersion, which tends to spread the signal out [36], as demonstrated by P. Singh et al. This makes it possible for solitons to retain their shape far and away and for long durations, placing them at the center of the analysis of nonlinear wave phenomena [41], as highlighted by V. E. Zakharov et al.

Soliton structures are not exotic phenomena [20], as demonstrated by A. Ja'afar Mohamad Jawad et al.; rather, they are quite functional in several areas of technology. In optical fibers, solitons [11] have been extensively studied by F. Sun et al. Demonstrating their potential for long-distance signal transmission without distortion, they can often be used

to transport information units over long distances without distortion. Taking advantage of the stability of the soliton [38], G. Hwang et al. have explored its applications in various fields, particularly in optical communication systems. In fluid dynamics, we see soliton-like behavior in shallow water waves. These works have helped in the explanation in the perception of such phenomena in the last few years in a way that may be considered useful for the knowledge of society especially on the nonlinear waves and the chaos in complicated system. For instance, Asghar Ali et al. [5] conducted a study on revealing the relationship between higher-order nonlinearity and soliton motion in the case of different types of chaotic patterns. The authors seek to understand different wave structures by investigating higher-order nonlinear wave terms. Asghar Ali et al. [3] later looked into the temporal and spatial behaviors of the solitons under various refractive index laws and argued that nonlinearity is significant in managing waves. Asghar Ali et al. [4] addressed bifurcation and chaos in fractional-order nonlinear systems, offering insights that directly complement our bifurcation and phase-space analysis, providing formation and stabilization of chaotic soliton structures, which contain approaches that are useful for investigating the effects of the external noises in this study. These investigations taken together, therefore, justify the use of fractional models and the methods of solution used in the present work. Where certain wave formations occur and travel over the water [2], G. Akram et al. have investigated their dynamics, contributing valuable insights into fluid dynamics and wave propagation. In some respects, solitary waves are the same as solitons, but they may not possess the same stability characteristics. Thus, solitons reveal the complexity of nonlinear reality and the capability of nonlinear and dispersive systems to produce localized, stable solutions [6], as explored by Y. Chahlaoui et al.

A dynamical system is a mathematical representation of a point in a specified space and how it changes in time according to given rules. Nonlinear dynamical systems are generally controlled by nonlinear equations, which means that the behavior of the system is prone to a sensitive response to initial conditions [15], as highlighted by A. Haluszczynski et al. This type of sensitivity can lead to dramatically different behaviors for only slightly different control parameters—the key characteristic of a chaotic system [28], is its extreme sensitivity to initial conditions, as discussed by Liu et al.

The specific equation is

$$\frac{\partial^{\delta} u}{\partial s^{\delta}} + Ku\frac{\partial^{\delta} u}{\partial y^{\delta}} - D\frac{\partial^{2\delta} u}{\partial y^{2\delta}} - mu + nu^{2} = 0. \tag{1}$$

Further, it shows that a specific dynamical system is nonlinear [26], as demonstrated by A. Khan et al., and it also represents an aspect of nonlinearity [8], as explored by N. Bhowmike et al.

Fractional calculus has a long history [16], with significant contributions from M. P. Lazarević et al. and H. Pratap et al. in advancing its applications and theoretical understanding. At the same time, it has recently been well-developed as a powerful tool for solving these problems. In contrast to integer-order derivatives, fractional derivatives, as explored by R. R. Khalil et al. [24], provides a powerful tool for modeling systems with memory and hereditary properties. This feature is useful to explain the occurrence of anomalous diffusion [27], as demonstrated by P. Kowalek et al., which gives a better picture of complex dynamics when particle transport does not conform to classical diffusion measurements.

The fractional diffusion-reaction method [12, 39], as explored by S. Yadav et al. and A. Ghafoor et al. provides a robust framework for addressing complex dynamics in various scientific fields. It is based on the diffusion-reaction concept, enhanced by fractional calculus, and provides an effective tool for investigating nonlinear dynamics, as shown by K. Saad et al. [34]. In such models, it becomes possible to describe the behavior of diffusive reaction processes for different substances, where the movement and interaction, or transformation of substances within the system, occur simultaneously, using fractional derivatives. This cluster-based aggregation method is especially effective if the processes in the system exhibit nonlocality or long-range interactions, which are typical for practical applications. This paper emphasizes the components of a general framework for applying the fractional diffusion-reaction process to model nonlinear dynamical systems, as highlighted by D. Yan et al. [40]. First, appropriate mathematical models have to be developed for the concrete processes that characterize the given system. This formulation may consist of setting the right order of boundaries and initial conditions and automatically setting the fractional order of the system. Second, the remaining fractional differential equations after the first stage are solved using a new set of precise numerical techniques.

R. Sharma et al. [35] discussed a numerical study for a nonlinear time-space fractional reaction-diffusion model of fourth-order and effectively captured the essence of the study. It highlighted the focus on a numerical approach, emphasizing the analysis of a nonlinear fractional reaction-diffusion model. The inclusion of "time-space fractional" signals a contemporary area of research dealing with nonlocal and memory effects, which are significant in modeling real-world phenomena. The mention of "fourth-order" added specificity, suggesting an exploration of higher-order complexities in the system.

Z. Hussain et al. have contributed significantly to bifurcation theory [1], advancing the understanding of critical transitions in nonlinear systems, and the periodic behavior of different systems is studied about how system parameters may cause a jump in the qualitative behavior as explored by S. M. Faris et al. [10]. What is more, as parameters are changed, a system might go through bifurcations—points of transition where a system's stability shifts, new equilibria appear, or periodic behavior can transition into chaos, as demonstrated by A. Jhangeer et al. [21], shedding light on the intricate dynamics of nonlinear systems. This understanding is important for interpreting possible critical transitions, for instance, when a fluid starts to become turbulent, and changes in population dynamics in the ecological models.

Chaos, which arises as a result of certain dynamic behaviors in nonlinear systems [37], has been extensively studied by W. Jia et al. providing valuable insights into complex dynamical phenomena associated with the fact that the behavior of the systems depends on small variations of the initial conditions. The difference of the minutest margin can have a massive impact; therefore, it is nearly impossible to predict the future over a long period. Studying chaos is important for many uses, including weather prediction and analyzing the fluctuations of the stock market. J. He et al. [17] applied advanced methodologies to uncover patterns and dynamics within financial systems.

Finally, the methodology must contain effective forecasting procedures that will use the characteristics of fractional calculus. Since forecasting in nonlinear systems is likely to be influenced by both the variability and the uncertainty inherent in the data sets, some complex tools may be needed. The combination of fractional models with machine learning or some other sophisticated statistical methods improves the predictions and allows car-

rying out a more extensive examination of the possible future state of the given system, as demonstrated by S. Raubitzek et al. [32]. That being said, this all-encompassing strategy has a great many potential applications of great potential importance. These models, for instance, can be employed in biological sciences to elaborate population dynamics or disease spreading, especially when global interactions are considered, for instance, migration of animals or disease contamination. Application of particles in environmental research, and particularly tracing of how contaminants or nutrients may move within ecosystems, can increase the knowledge of ecosystem function and aid in the direction of management. Fractional modeling methods can also be applied to engineering to enhance the design and analysis of the compound characteristics of materials. For example, viscoelasticity [31], as studied by C. Meral et al., demonstrates the practical application of fractional models in capturing the complex behavior of materials with memory effects.

Consequently, the inclusion of fractional calculus in the analysis of nonlinear dynamical systems modeling and forecasting is a major evolution in contemporary analytical tools [23], which has significantly advanced the study of complex systems by providing deeper insights into their underlying dynamics. The specification of the mathematical model, the numerical solution, and the efficient realization of accurate forecasts enhance the understanding of complex nonlinearity and provide better results in prediction. The following subtopics describe the theoretical background of fractional calculus, the use of the fractional diffusion-reaction method, and sample applications of fractional calculus for analyzing nonlinear dynamical systems.

2 The proposed method's outline

2.1 Modified Liouville derivative

The modified Riemann-Liouville derivative is one of the more effective approaches for studying nonlinear dynamic systems. At present, it remains an essential tool in solving a variety of problems in various branches of science and engineering because of its capability to address the effects of memory and nonlocality in observed behaviors.

The characteristics of the modified Riemann-Liouville derivative used in this work are as follows. The modified Riemann-Liouville derivative of order ξ by Jumarie is given by Khalil et al. [25]:

$$D_z^{\xi} h(z) = \frac{1}{\Gamma(-\xi)} \int_0^z (z-\eta)^{-\mu-1} [h(\xi) - h(0)]\, d(\eta). \tag{2}$$

$\mu < 0$,

$$D_z^{\xi} h(z) = \frac{1}{\Gamma(1-\xi)} \frac{d}{dz} \int_0^z (z-\eta)^{-\mu} [h(\xi) - h(0)]\, d(\eta). \tag{3}$$

$0 < \mu < 1$,

$$D_z^{\xi} \int_0^{\xi} h(z) = (h^n(z))^{\xi-n}. \tag{4}$$

$n \le \mu \le n+1$, $n \ge 1$,
where, the function $h : R \to R$ is constant. Further, the following three useful formulas for the fractionally modified Derivative of Riemann-Liouville are given below, which encom-

passes the given nonlinear ODE in the form:

$$D_s^{\xi} z^{\lambda} = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda-\xi)} z^{\lambda-\xi}, \lambda > 0. \tag{5}$$

$$D_z^{\xi}(\alpha(z)\beta(z)) = \beta(z)D^{\xi}\alpha(z) + \alpha(z)D^{\xi}\beta(z). \tag{6}$$

$$D_z^{\xi}(h(z)\alpha(z)) = h'_{\alpha}(\alpha)D_z^{\xi}\alpha(z) = D_{\alpha}^{\xi}h(\alpha)(\alpha'_z)^{\xi}, \tag{7}$$

which are immediate effects of the $d^{\xi} z(s) = \Gamma(1+\xi)d_z(s)$.

The generalized Riccati equation mapping method, a comprehensive framework, is described in detail to solve the specified NLPDEs.

$$M(u, u_y, u_s, u_{yy}, u_{ss}, \ldots) = 0, \tag{8}$$

where M is the partial derivative of $u(y,s)$ concerning some parameters for which only y and s are denoted in this work. As a result, the electromagnetic wave solution is obtained by the transmutation approach. By transformation, the generalized Riccati equation mapping method gives the following equation:

$$u(y,s) = U(\xi), \qquad \xi = \frac{y^{\delta}}{\Gamma(1+\delta)} - \frac{\lambda s^{\delta}}{\Gamma(1+\delta)}, \tag{9}$$

Using the above transformation, we convert Eq. (1) into a nonlinear ODE.

$$G(U, U', U'', \ldots) = 0, \tag{10}$$

where G is polynomial in $U(\xi)$, the U is incorporated such that the first derivative $U' = \frac{dU}{d\xi}$, $U'' = \frac{d^2U}{d\xi^2}$, and so on.

$$-\xi U' + KUU' - EU'' - mU + nU^2 = 0. \tag{11}$$

By setting $U' = 0$, we gain a focused view of the system's equilibrium states, simplified bifurcation analysis, and direct insight into the nonlinear interaction that governs stability. This simplification highlights stable and unstable points. We can directly identify equilibrium points and their stability condition. It also determines the parameter-driven behavior shift and nonlinear dynamics. It becomes easier to analyze how changing parameters affect stability, helping us understand bifurcation points, and by decomposing the expansion to separate the nonlinear terms, we can numerically investigate self-interaction and the possibility of changes in the stability or behavior of the system. Then the terms involving U' (i.e., $-\xi U'$ and KUU') vanish, and we obtain

$$-EU'' - mU + nU^2 = 0. \tag{12}$$

This is now a second-order differential equation with only two variables, U and U'':

$$EU'' = -mU + nU^2. \tag{13}$$

By rearranging, we get

$$U'' = \frac{-m}{E}U + \frac{n}{E}U^2. \tag{14}$$

Suppose that the following is the form of the result of Eq. (10):

$$U(\xi) = \sum_{i=0}^{n} m_i W^i(\xi), \tag{15}$$

where

$$W'(\xi) = \mu + \nu W(\xi) + \lambda W^2(\xi). \tag{16}$$

It is important to notice that in this case, μ, ν, and λ are constant.

Using the Maple program to solve the algebraic equations given above, the following trivial results can be obtained:

$$e0 := -E\mu\nu + K\mu m_0 + nm_0^2 - \eta\mu - mm_0 = 0,$$

$$e1 := -2E\lambda\mu - E\nu^2 + K\mu m_1 + K\nu m_0 + 2nm_0m_1 - \eta\nu - mm_1 = 0,$$

$$e2 := -3E\lambda\nu + K\lambda m_0 + K\nu m_1 + nm_1^2 - \eta\lambda = 0,$$

$$e3 := -2E\lambda^2 + K\lambda m_1 = 0.$$

Set 1:

$$m_0 = \frac{2(\frac{1}{2}\nu + \frac{1}{2}\sqrt{(\nu^2 - 4\lambda\mu)})E}{K}, \quad m_1 = \frac{2E\lambda}{K}. \tag{17}$$

Set 2:

$$m_0 = \frac{2(\frac{1}{2}\nu - \frac{1}{2}\sqrt{(\nu^2 - 4\lambda\mu)})E}{K}, \quad m_1 = \frac{2E\lambda}{K}. \tag{18}$$

2.1.1 Case 1

If $D = \nu^2 - 4\,\mu\lambda \geq 0$ and $\mu\lambda \neq 0$ OR $\nu\lambda \neq 0$, then:

$$U_{1,1}(\xi) = \frac{(\nu \pm \sqrt{\nu^2 - 4\mu\lambda})E}{K} - \frac{\sqrt{D}tanh(\frac{\sqrt{D}(\xi)}{2})E}{K} - \frac{\nu E}{K}, \tag{19}$$

$$U_{1,2}(\xi) = \frac{(\nu \pm \sqrt{\nu^2 - 4\mu\lambda})E}{K} - \frac{\sqrt{D}coth(\frac{\sqrt{D}(\xi)}{2})E}{K} - \frac{\nu E}{K}, \tag{20}$$

$$U_{1,3}(\xi) = \frac{(\nu \pm \sqrt{\nu^2 - 4\mu\lambda})E}{K} - \frac{\sqrt{D}tanh(\sqrt{D}(\xi))E \pm \iota sech(\sqrt{D}(\xi))E}{K} - \frac{\nu E}{K}, \tag{21}$$

$$U_{1,4}(\xi) = \frac{(\nu \pm \sqrt{\nu^2 - 4\mu\lambda})E}{K} - \frac{\sqrt{D}E(tanh(\frac{\sqrt{D}(\xi)}{4}) + coth(\frac{\sqrt{D}(\xi)}{4}))}{2K} - \frac{\nu E}{K}, \tag{22}$$

$$U_{1,5}(\xi) = \frac{(\nu \pm \sqrt{\nu^2 - 4\mu\lambda})E}{K} + \frac{2E\lambda}{K}[\frac{2\mu cosh(\sqrt{D}(\xi))}{\sqrt{D}sinh(\sqrt{D}(\xi)) - \nu cosh(\sqrt{D}(\xi)) \pm \iota\sqrt{D}}], \tag{23}$$

$$U_{1,6}(\xi) = \frac{(\nu \pm \sqrt{\nu^2 - 4\mu\lambda})E}{K} + \frac{2E\lambda}{K}[\frac{2\mu sinh(\sqrt{D}(\xi))}{\sqrt{D}cosh(\sqrt{D}(\xi)) - \nu sinh(\sqrt{D}(\xi)) \pm \iota\sqrt{D}}]. \tag{24}$$

2.1.2 Case 2

If $D = \nu^2 - 4\,\mu\lambda < 0$ and $\mu\lambda \neq 0$ OR $\nu\lambda \neq 0$, then:

$$U_{1,7}(\xi) = \frac{(\nu \pm \sqrt{\nu^2 - 4\mu\lambda})E}{K} + \frac{\sqrt{-D}tan(\frac{\sqrt{-D}(\xi)}{2})E}{K} - \frac{\nu E}{K}, \tag{25}$$

$$U_{1,8}(\xi) = \frac{(\nu \pm \sqrt{\nu^2 - 4\mu\lambda})E}{K} - \frac{\sqrt{-D}cot(\frac{\sqrt{-D}(\xi)}{2})E}{K} - \frac{\nu E}{K}, \tag{26}$$

$$U_{1,9}(\xi) = \frac{(\nu \pm \sqrt{\nu^2 - 4\mu\lambda})E}{K} + \frac{\sqrt{-D}tan(\sqrt{-D}(\xi))E \pm \iota sec(\sqrt{-D}(\xi))E}{K} - \frac{\nu E}{K}, \tag{27}$$

$$U_{1,10}(\xi) = \frac{(\nu \pm \sqrt{\nu^2 - 4\mu\lambda})E}{K} + \frac{\sqrt{-D}E(tan(\frac{\sqrt{-D}(\xi)}{4}) - cot(\frac{\sqrt{-D}(\xi)}{4}))}{2K} - \frac{\nu E}{K}, \tag{28}$$

$$U_{1,11}(\xi) = \frac{(\nu \pm \sqrt{\nu^2 - 4\mu\lambda})E}{K} - \frac{2E\lambda}{K}[\frac{2\mu cos(\sqrt{-D}(\xi))}{\sqrt{-D}sin(\sqrt{-D}(\xi)) + \nu cos(\sqrt{-D}(\xi)) \pm \iota\sqrt{-D}}], \tag{29}$$

$$U_{1,12}(\xi) = \frac{(\nu \pm \sqrt{\nu^2 - 4\mu\lambda})E}{K} - \frac{2E\lambda}{K}[\frac{2\mu sin(\sqrt{-D}(\xi))}{\sqrt{-D}cos(\sqrt{-D}(\xi)) - \nu sin(\sqrt{-D}(\xi)) \pm \iota\sqrt{-D}}]. \tag{30}$$

2.1.3 Case 3

(a) The solution for $\mu = \nu = 0$ and $\lambda \neq 0$

$$U_{1,13}(\xi) = \frac{(\nu \pm \sqrt{\nu^2 - 4\mu\lambda})E}{K} \pm \frac{2E\lambda}{K\lambda(\xi)}. \tag{31}$$

(b) The solution for $\mu = \nu = 0$ and $\lambda = \frac{\nu^2}{4\mu}$

$$U_{1,14}(\xi) = \frac{(\nu \pm \sqrt{\nu^2 - 4\mu\lambda})E}{K} - \frac{4E\lambda\mu(\nu(\xi) + 2)}{K\nu^2(\xi)}. \tag{32}$$

2.1.4 Case 4

(a) The solution for $\lambda = 0$, $\mu = K\nu$ is

$$U_{1,15}(\xi) = \frac{(\nu \pm \sqrt{\nu^2 - 4\mu\lambda})E}{K} + \frac{2E\lambda(e^K(\xi) - K)}{K}, \tag{33}$$

(b) The solution for $\lambda = \nu \neq 0$ and $\mu = 0$ is

$$U_{1,16}(\xi) = \frac{(\nu \pm \sqrt{\nu^2 - 4\mu\lambda})E}{K} - \frac{2\nu\rho E}{K(e^{-}\nu(\xi) + \rho)}. \tag{34}$$

$$U_{1,17}(\xi) = \frac{(\nu \pm \sqrt{\nu^2 - 4\mu\lambda})E}{K} - \frac{2\nu E e^{\nu}(\xi)}{K(e^{\nu}(\xi) + \rho)}. \tag{35}$$

3 Physical discussion

In this article, we use Mathematica to edit two-dimensional (2*D*) and three-dimensional (3*D*) graphs to illustrate the traveling wave solutions found in the GREMM equation and highlight the features of the different wave solutions. Only a few of the many methods for

understanding and depicting interconnected physical processes are shown in Figs. 1–7. For every result, we have improved several soliton profiles using different exact values of the preexisting parameters. Here is the physical representation of Set 1 and Set 2. The graph shows the solution of $U(x,t)$ in two and three dimensions, respectively. Solitary waves are forms of waveforms that take a solitary configuration and move at certain velocities at which they do not change their shape. They occur in nonlinear systems and do not lose their form when interfering with other waves.

Here, the removal of single waves, as represented by Eq. (19), falls under traveling wave solutions. The graphical interpretation of Fig. 1 is described with the following velocity values: $\nu = 1.2$, $\mu = 1.1$, $\lambda = 1.3$, $E = 3$, $D = 0.53$, and $K = 2$. The graph shows the solution of $U_{1,1}(x,t)$ in $2D$ and $3D$, respectively. The removal of single waves, as seen in Eq. (20), falls under traveling wave solutions. The graphical interpretation of Fig. 2 is described with the following velocity values: $\nu = 1.2$, $\mu = 1.1$, $\lambda = 1.3$, $E = 3$, $D = 0.53$, and $K = 2$. The graph shows the solution of $U_{1,2}(x,t)$ in $2D$ and in $3D$, respectively.

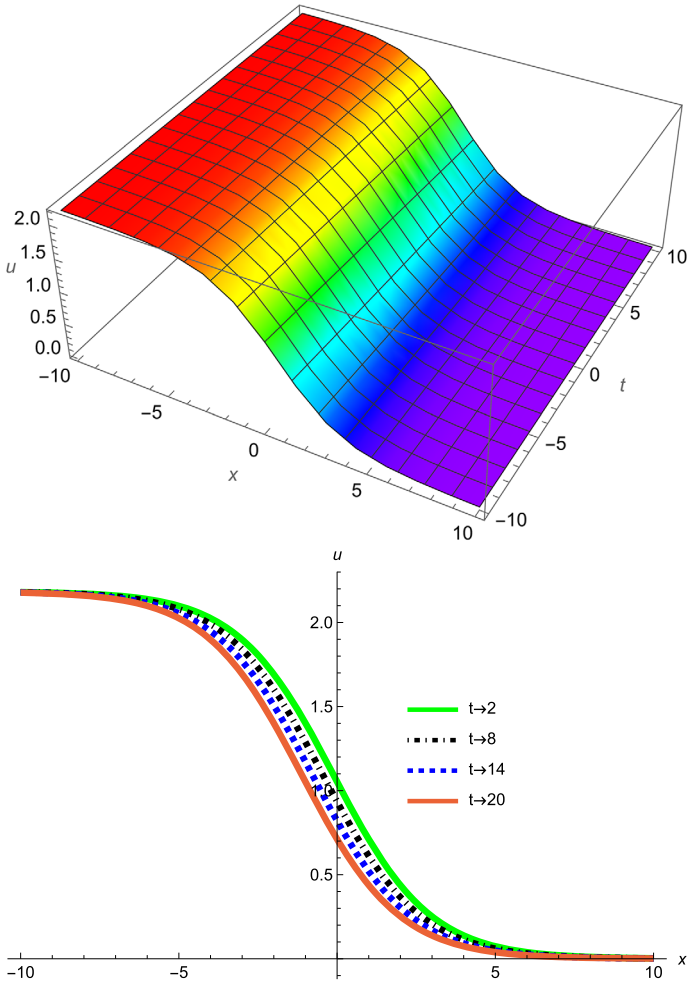

Figure 1 The solution of $U_{1,1}(x,t)$ represents the motion of a wave through time in two- and three-dimensional planes. Here, x can be considered the normalized distance that the wave has just passed, and t is the normalized time (t). Over time, the behavior of the wave is altered at various state stages, such as $t = 2, 8, 14, 20$

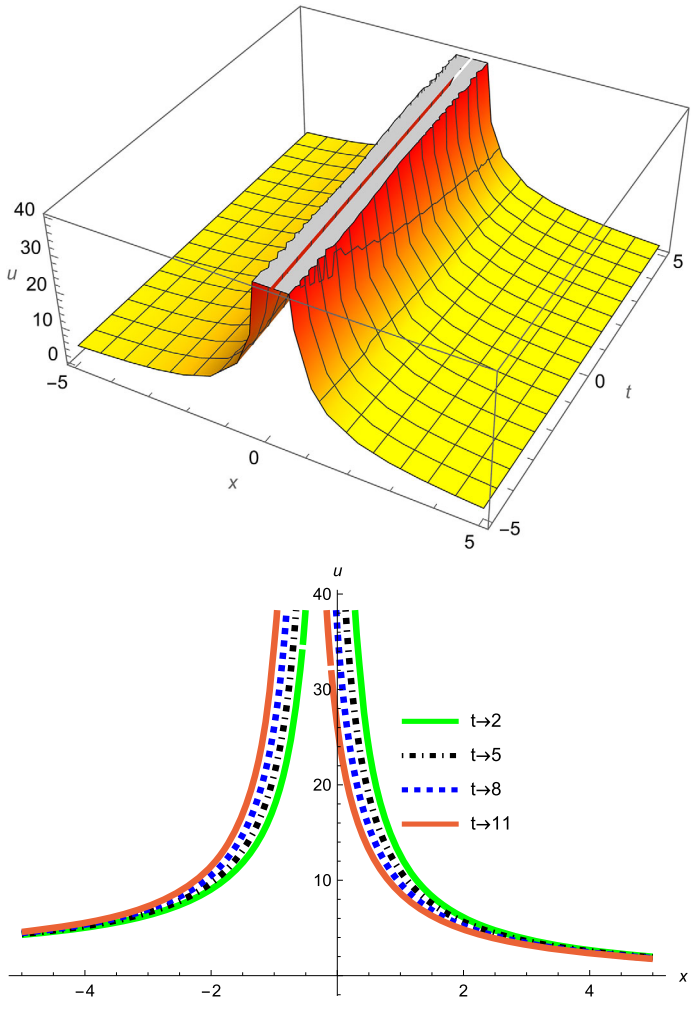

Figure 2 The solution of $U_{1,2}(x,t)$ represents the motion of a wave traveling through time in two- and three-dimensional planes. Here, x can be considered the normalized distance that the wave has just passed, and t is the normalized time (t). Over time, the behavior of the wave is altered at various state stages, such as $t = 2, 8, 14, 20$

Figure 3 demonstrates the dynamics of the wave in $2D$ and $3D$, represented by $U_{1,5}(x,t)$ as described in Eq. (23). Its parameters are as follows: $\nu = 1.2$, $\mu = 1.1$, $\lambda = 1.3$, $E = 3$, $D = 0.53$, and $K = 2$ over intervals $-10 < x < 10$ and $-10 < t < 10$. The wave's amplitude changes over time, and its consistent structure suggests a solitary or soliton wave, often characterized by stability due to a balance between nonlinearity and dispersion. $3D$ Plot: This visual shows how the wave travels through time when the wave persists over time toward the next figure, where the wave moves along the x-axis to capture time (as indicated by the increases in t). The peaks and troughs of the wave highlight variations in amplitude, possibly pointing to the wave's soliton nature, as it maintains its structure. $2D$ Plot: This set of cross sections illustrates the wave amplitude at different instants of measurement, as illustrated in the figure below. Thus, it is possible to observe a uniform wave pattern throughout different time divisions, such as ($t = 2,\ 10,\ 18,\ 26$). Indeed, as shown in Fig. 3, where one can see the wave amplitude and phase at ($t = 2,\ 8,\ 14,\ 20$), it may be nondispersive, which is a common feature of solitons.

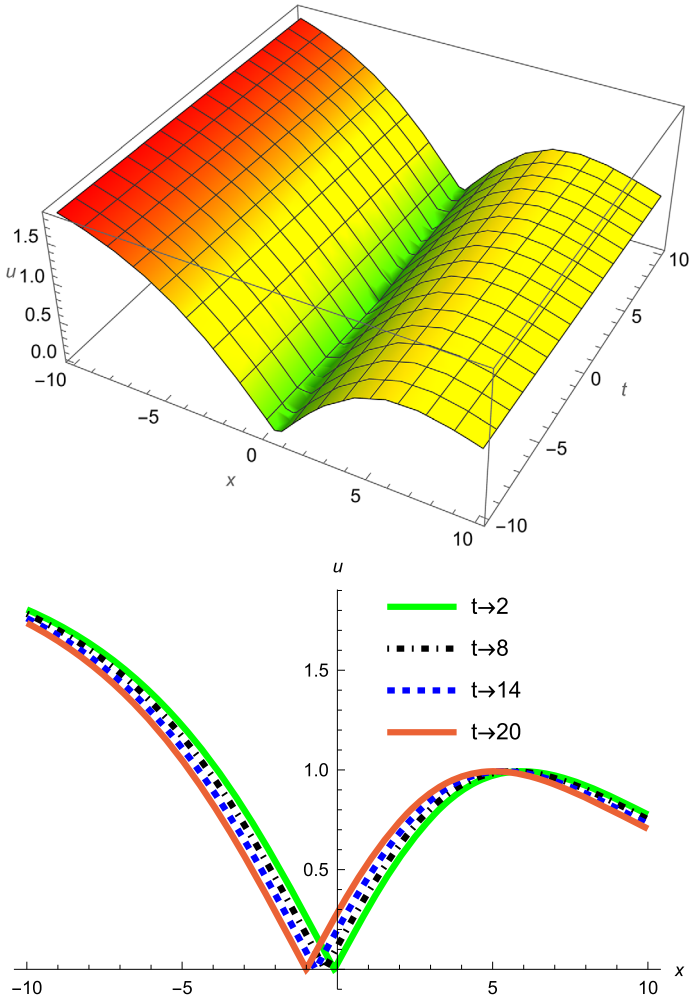

Figure 3 The solution of $U_{1,5}(x,t)$ represents the motion of a wave traveling through time in two- and three-dimensional planes. Here, x can be considered the normalized distance that the wave has just passed, and t is the normalized time (t). Over time, the behavior of the wave is altered at various state stages, such as $t = 2, 10, 18, 26$

Figure 4 shows a solitary (soliton) wave propagating in space and time: 3D view and 2D view with the following parameters: $\mu = 1.5$, $\nu = 2.5$, $\lambda = 1.3$, $D = -1.55$, $K = 2$ over intervals $-5 < x < 5$ and $-5 < t < 5$, Eq. (25). This wave keeps a stable and localized maximum because the nonlinearity that compresses the wave and dispersion that extends the wave is balanced so that there is a stable "packet" of energy that travels. In the 3D view, we have the wave profile when it is in the spatial axis, and while in the 2D view, we have wave profile cross sections at different times (for example, $t = 0.1, 0.3, 0.5, 0.7$). The results plotted demonstrate the stability and symmetry of this function. Solitons of this nature are typical of the system mentioned above, including optical fibers, water waves, and plasma, where undistorted signal transmission is desirable. Figure 5 provides an impression of a wave movement; the billowing form in 2D and 3D that is owned by parameters is dominated by, $\mu = 1.5$, $\nu = 2.5$, $\lambda = 1.3$, $D = -1.55$, and $K = 2$ over intervals $-5 < x < 5$ and $-5 < t < 5$. The change or oscillation rate portraying the flexing of the wave, represented by the curvature of the wave, is the activity level or amplitude/intensity of energy. Probably, when

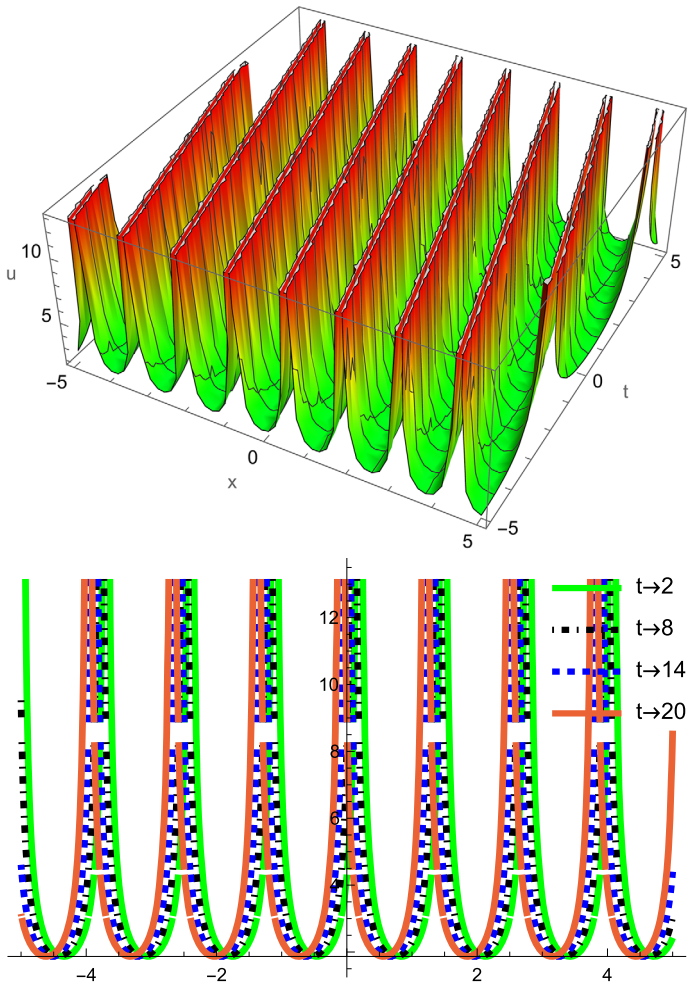

Figure 4 The solution of $U_{1,7}(x,t)$ represents motion of a wave traveling through time in two- and three-dimensional planes. Here, x can be considered the normalized distance that the wave has just passed, and t is the normalized time (t). Over time, the behavior of the wave is altered at various state stages, such as $t = 2, 8, 14, 20$

writing the mentioned equation, which is known as Eq. (26), one solves a nonlinear partial differential equation that defines the shape of a solitary or soliton wave and studies how the form evolves in time and how the wave retains or alters its profile due to the combination.

The first plot demonstrates a $3D$ travel wave in which the wave moves along the x-axis as time increases along the t-axis. The bottom plot represents a $2D$ profile concerning the wave amplitude at a particular time, for instance, $t = 2,\ 8,\ 14,\ 20$. The curved line in each picture illustrates the shape of the wave at that particular time, which was described by the function to represent the evolution of the signal. Soliton waves do not change shape when traveling, and in this case, the solitary wave retains its shape. Figure 6 provides information concerning the movement of a wave in space and time, or space and time relationships. The $3D$ plot at the top shows the wave's evolution in space and time using the following parameters: $\mu = 1.5$, $\nu = 2.5$, $\lambda = 1.3$, $D = -1.55$, $K = 2$ over intervals $-5 < x < 5$ and $-5 < t < 5$. The following $2D$ plots depict the wave's amplitude ratings over time-related intervals.

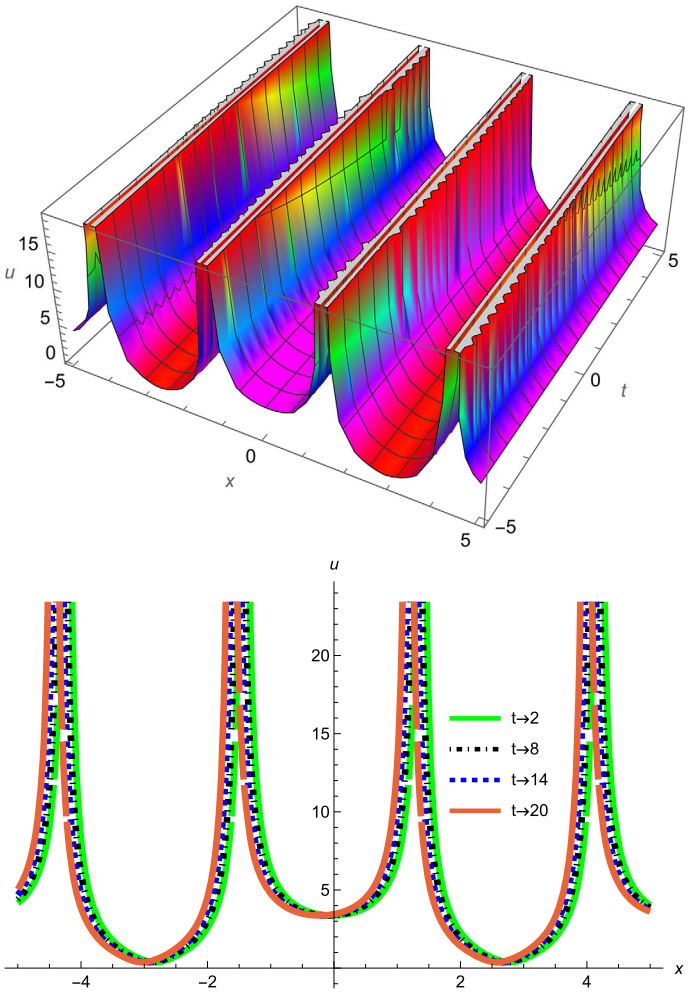

Figure 5 The solution of $U_{1,9}(x,t)$ represents motion of a wave traveling through time in two- and three-dimensional planes. Here, x can be considered the normalized distance that the wave has just passed, and t is the normalized time (t). Over time, the behavior of the wave is altered at various state stages, such as $t = 2, 8, 14, 20$

Every plot represents the condition of the wave amplitude at some given time to represent the alteration of the amplitude.

3D Surface Plot: The wave amplitude of $U_{1,11}(x,t)$. The function $u(x,t)$ in Fig. 6 also evolves as it occurs along the spatial direction x over time t. This plot reveals more information as to how the crests and troughs change, as well as providing a view of how the wave behaves and moves as time progresses.

2D Line Plot: In this section, the wave amplitude at different times is illustrated by capturing a picture of the wave shape at a given instance ($t = 0.1,\ 0.4,\ 0.7,\ 1$). This entitles one to view the variation of the wave structure by amplitude and shape at these particular time instants. Every curve in the given plot indicates the wave profile at a particular time thus giving information about the wave as it goes on. This analysis is within the framework of Eq. (29).

Figure 7 provides information concerning the movement of a wave in space and time, or space and time relationships. The 3D plot at the top shows the wave's evolution in space

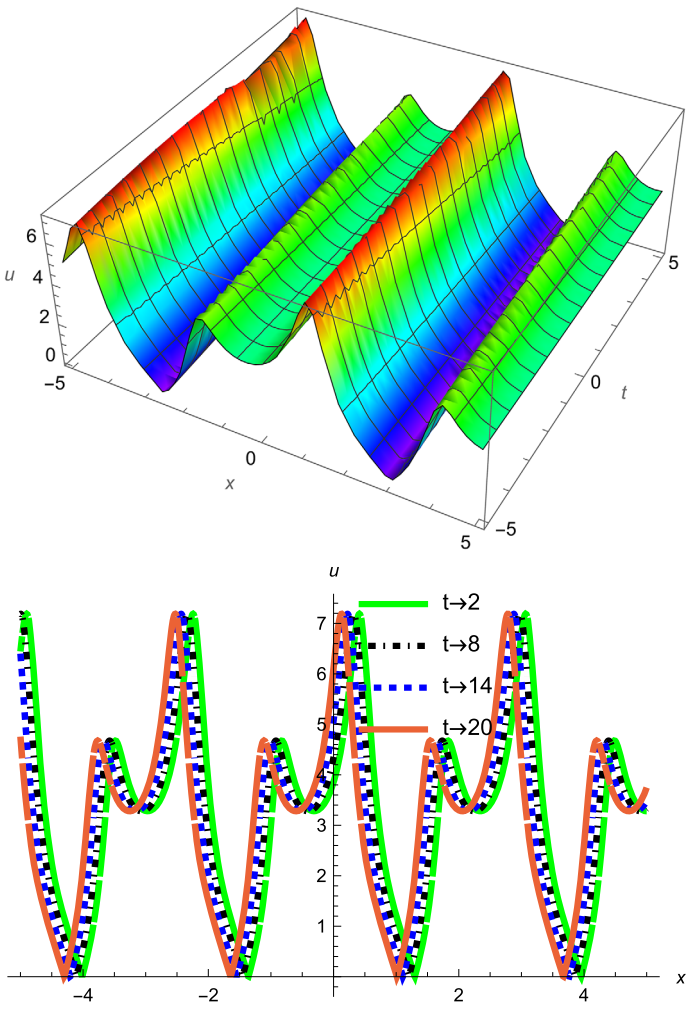

Figure 6 The solution of $U_{1,11}(x,t)$ represents motion of a wave traveling through time in two- and three-dimensional planes. Here, x can be considered the normalized distance that the wave has just passed, and t is the normalized time (t). Over time, the behavior of the wave is altered at various state stages, such as $t = 2, 8, 14, 20$

and time using the following parameters: $\mu = 1.5$, $\nu = 2.5$, $\lambda = 1.3$, $D = -1.55$, $K = 2$ over intervals $-5 < x < 5$ and $-5 < t < 5$. The following $2D$ plots depict the wave's amplitude ratings concerning time-related intervals. Every plot represents the condition of the wave amplitude at some given time to represent the alteration of the amplitude. $3D$ Surface Plot: The wave amplitude of $U_{1,12}(x,t)$. The function $u(x,t)$ in Fig. 7 also evolves as it occurs along the spatial direction x over time t. This plot reveals more information as to how the crests and troughs change and provides a view of how the wave behaves and moves as time progresses.

$2D$ Line Plot: In this section, the wave amplitude at different times is illustrated by capturing a picture of the wave shape at a given instance ($t = 2,\ 8,\ 14,\ 20$). This entitles one to view the variation of the wave structure by amplitude and shape at these particular time instants. Every curve in the given plot indicates the wave profile at a particular time thus giving information about the wave as it goes on. This analysis is within the framework of Eq. (30).

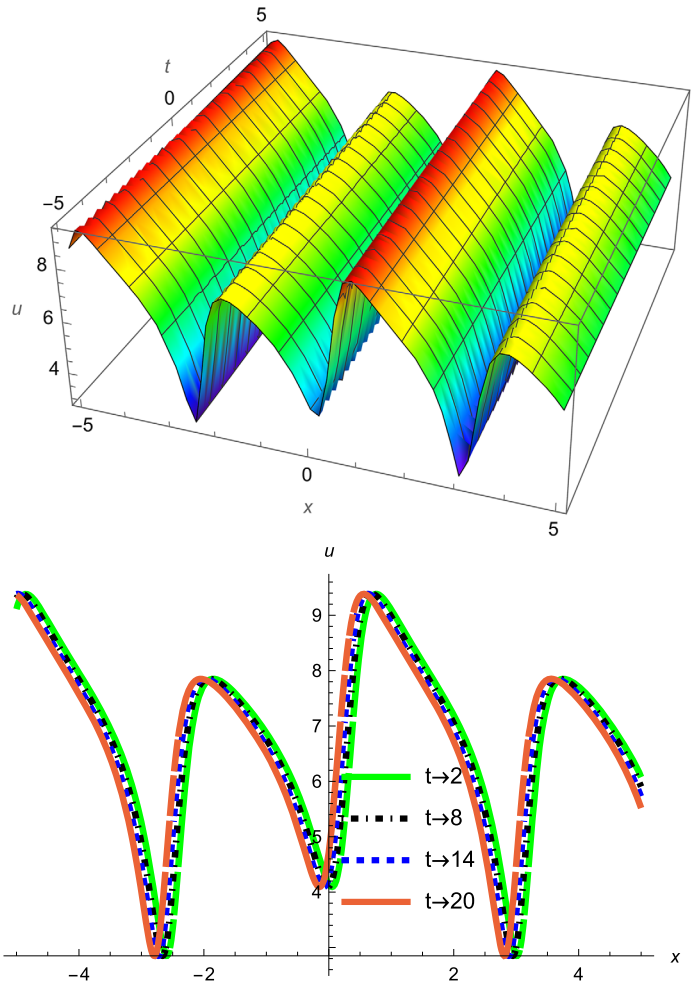

Figure 7 The solution of $U_{1,12}(x,t)$ represents the motion of a wave through time in two- and three-dimensional planes. Here, x can be considered the normalized distance that the wave has just passed, and t is the normalized time (t). Over time, the behavior of the wave is altered at various state stages, such as $t = 2, 8, 14, 20$

4 Phase representations and bifurcation behavior

Bifurcation analyses are key to our work with nonlinear dynamical systems as they offer a simple representation of how system dynamics changes with the parameters. In this work, they assist with defining change points, the precise transition as traveling waves, periodic waves, or the appearance of a new stability, within complex systems. For some of this work, it suffices to plot fixed points against particular parameters to identify regions of multistability and chaoticities, which are useful for the comprehension of the dynamics of the system. When solving nonlinear partial differential equations important for my investigation, bifurcation diagrams help to identify how the change of a small parameter can result in a drastic change of characteristics. This insight is important in the identification of the future state of the system and its various turbulent characteristics. In technological physics, such as hydrodynamics, or in biological physics, such as population dynamics in models of ecology. Furthermore, fractional calculus used in this work enriches the interpretation of these diagrams. This is through capturing factors such as memory and

history. This allows the analysis of the system behavior regarding alterations of parameters to be more accurate. In general, bifurcation diagrams are one of the most important instruments in our research, since they reveal crucial aspects of the behavior of nonlinear systems and help to make predictions and analysis in a wide range of applications.

$$\begin{cases} \frac{dU}{d\xi} = P, \\ \frac{dP}{d\xi} = A_0 U + A_1 U^2. \end{cases} \tag{36}$$

The bifurcation analysis is used to show the switch between other equilibria and stability of the system where $A_0 = -m$ and $A_1 = n$. There are two equilibrium points in the system (36) mentioned above, which are as follows:

$U_1 = (E_1, 0)$ and $U_2 = (E_2, 0)$.

Here, E_1 and E_2 values are defined as follows:

$E_1 = (0, 0)$ and $E_2 = (1, 0)$.

According to the concept of dynamical structure, we present several strands of phase portrait trajectories at the qualitatively different phases to investigate the qualitative behavior of the system.

Nonlinear Periodic Trajectory (*NPT*): In this system, a closed-loop trajectory is referred to as $NHPT(x, y)$. It has a periodic character and can be referred to as $NHPT(x, y)$. This is a trend that means that undergoes oscillation or cycles around the equilibrium point. There is an indication that the system cycles a condition which suggests that it cycles under certain conditions.

Nonlinear Homoclinic Periodic Trajectory (*NHPT*): This type of trajectory will be referred to as $NHPT(x, y)$. $NHPT(x, y)$ starts and finishes at the same point of stability and so does not connect to other trajectories, meaning it is a phase loop of change that brings the system back to an equilibrium state in the phase space.

4.1 Case 1

• $A_0 > 0, A_1 > 0$.

Dynamical Structure: In this dynamical system, there are two equilibrium points, probably a saddle point, and one is the center point.

Behavior:

Center Point: The closed, periodic orbits (the blue and green curves) speak of a center point where the attached stable, bounded orbits revolve around the equilibrium point.

Saddle Point: It illustrates the red curves from where trajectories approach the given point and then diverge in different directions, which means that it is a saddle point.

Graph Interpretation: This is illustrated in Fig. 8(a), whereas within the phase portrait, there is a saddle point which is located at the center, and there are trajectories entering and exiting in different directions.

4.2 Case 2

• $A_0 > 0, A_1 < 0$.

Dynamical Structure: There is now a change in the dynamics of the system, but two equilibrium points also exist in it. One acts as a center point and the other as a saddle point.

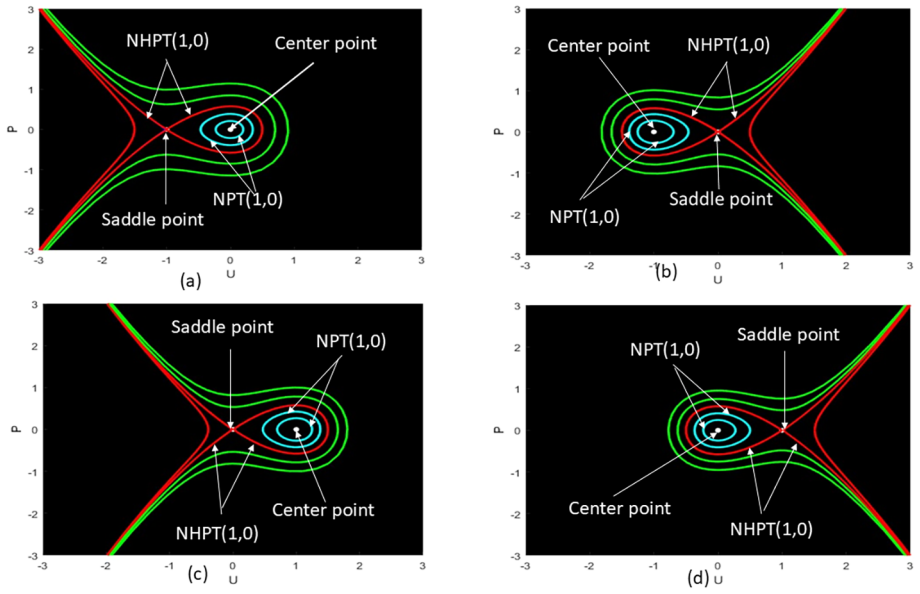

Figure 8 Bifurcation diagrams of Eq. (1)

Behavior: The saddle point is represented at $F_0(A_0, 0)$, where the trajectories converge to it and then move away from it. For the center points, there are periodic orbits, which show stability and boundedness of the solution movement of points around the center locations.

Graph Interpretation: This must be depicted in Fig. 8(b), where the trajectories are more complex with saddle and center points.

4.3 Case 3

• $A_0 > 0, A_1 < 0$.

Dynamical Structure: The conditions mentioned above satisfy at least two equilibrium points of the system: one is a center point, while the other is a saddle point.

Behavior: This implies that for center points, it is permitted to have circular or periodic motion trajectories that are not going off to infinity. The saddle point has directions that come closer and then turn in opposite direction.

Graph Interpretation: This case is shown in Fig. 8(c), where there is a central point with circular trajectories and a saddle point close to it that disturbs the motion.

4.4 Case 4

• $A_0 < 0, A_1 < 0$.

Dynamical Structure: When both $A_0 < 0$, and $A_1 < 0$. The dynamical system exhibits significant changes in its behavior. However, there are also two equilibrium points. One acting as a center point and the other as a saddle point.

Behavior:

Saddle Point: The saddle point remains. However, the paths that may get to it and then continue from it can be more complicated and capricious, where it can include both linear pathways that lead toward such structures or trajectories that recede from them.

Center Point: If center points exist, then the system can have stable points of the period around a center point. However, these orbits might be less stable and therefore could exhibit chaotic dynamics.

Graph Interpretation:

Graphs may illustrate more complex behavior of the given function – variations near and around the saddle and center points; see Fig. 8(d). There may be additional separatrices that make the unstable manifolds more apparent around the saddle points.

5 Chaotic and quasi-periodic behaviors

In this section, we will detail the properties of chaotic/quasi-periodic motions of the perturbed dynamical systems. In particular, system (36) will be examined with the perturbation term $f_0 cos(\omega t)$ added, yielding the following form:

$$\begin{cases} \frac{dU}{d\xi} = P, \\ \frac{dP}{d\xi} = A_0 U + A_1 U^2 + f_0 cos(\omega t), \end{cases} \tag{37}$$

where ω is the constant of the frequency of the external disturbance, and f_0 is the constant of the magnitude of the external periodic disturbance. Here, we employ several numeric approaches to analyze each kind of quasi-periodic and chaotic behavior of the system (37), as analyzed by Y. Jin et al. [22], providing crucial insights into the unpredictable dynamics inherent in nonlinear systems. Nine independent parameters make up the perturbed dynamical system: I, a, b, c, d, A, B, C, and D. It is quite difficult to analyze the systems by changing every parameter because there are so many of them. To make the study simpler, we thus took into account particular values of the parameters, such as $I = 100$, $a = 2.7$, $b = 150$, $c = -0.00025$, $d = 0.0005$, $A = 0.5$, $B = -6$, and $D = 4$. To observe the chaotic and quasi-periodic aspects in the dynamical system, the final two parameters, ω and f_0, are essential systems (37).

5.1 Quasi periodic oscillations

Case 1: $f_0 = 0$:

In Fig. 9(a), (b), and (c), we can observe the quasi-periodicity of the perturbed dynamical system (37) for $\omega = 0.64$. The time series plot and phase portrait are shown in Fig. 9 (a), (b), and (c). In a phase portrait, instead of identifying one trajectory that overlaps another, the usual paths are shown in the phase plane. Both the amplitude of the curve in the time series plot and the period, Ti, are constant; notice that they were compared with the period T_{i+1}. They are slightly less. The behavior of the system described above proves that it is nearly in a quasi-periodic cycle, comparing the results of this section to those presented in [7] by M. Bahar et al., highlighting the key similarities and differences in the behavior and dynamics of the studied systems. Therefore, we have established that the production of quasi-periodic motions depends on f_0 and is affected in the same way.

5.2 Quasi periodic oscillations

Case 2: $f_0 \neq 0$:

In this subsection, the quasi-periodic behavior of the perturbed dynamics given by the system (37) is investigated, with $f_0 = -1.49$ and $\omega = 0.64$ chosen as parameters. Using the

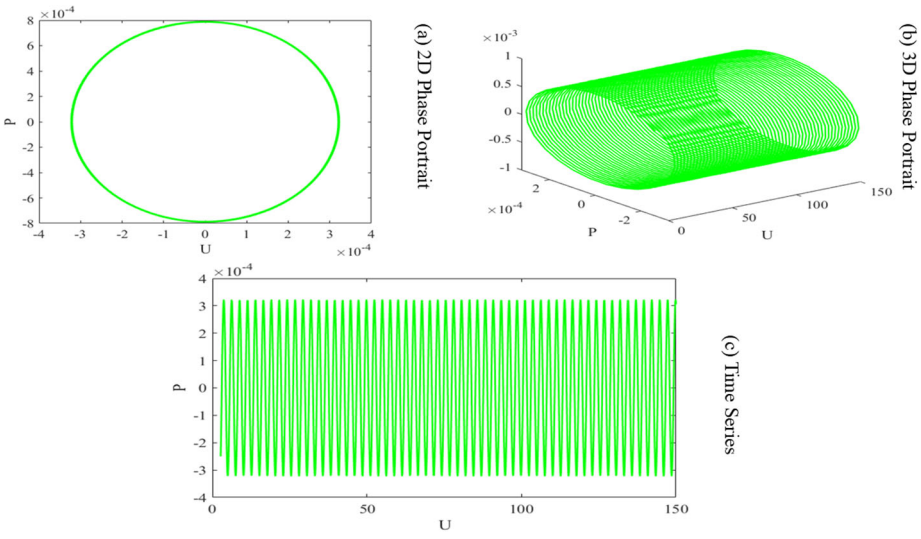

Figure 9 (a) 2D Phase portrait, (b) 3D Phase portrait, (c) Time series of the NLDS Eq. (1), for $f_0 = 0$ and $\omega = 0.64$

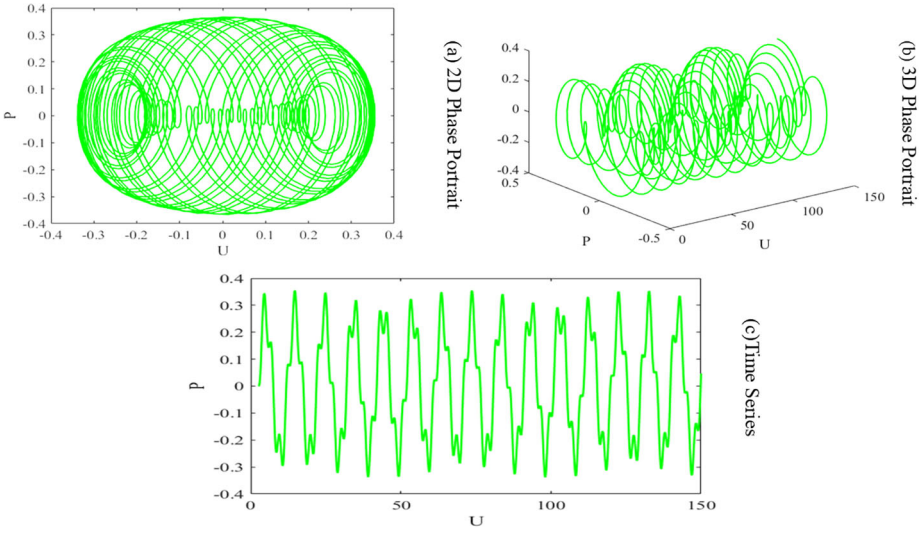

Figure 10 (a) 2D Phase portrait, (b) 3D Phase portrait, (c) Time series of the NLDS Eq. (1), for $f_0 = -1.49$ and $\omega = 0.64$

same parameters marked in Fig. 9, the results are presented in Fig. 10 (a), (b), and (c), representing the time series and phase portrait, respectively.

The phase portrait is similar to Fig. 10 but includes an additional normal trajectory drawn over it. Within a clearly defined time interval, the time series generates packets with amplitudes similar to others of the same kind. These graphs show that the system is quasi-periodic in its behavior.

Different values of f_0 and ω have produced varying results in this study, despite the use of identical parameter values in Fig. 9 and 10. This variation reveals patterns and shapes in the time series and phase portrait, demonstrating how sensitive the system is to the changes in these parameters.

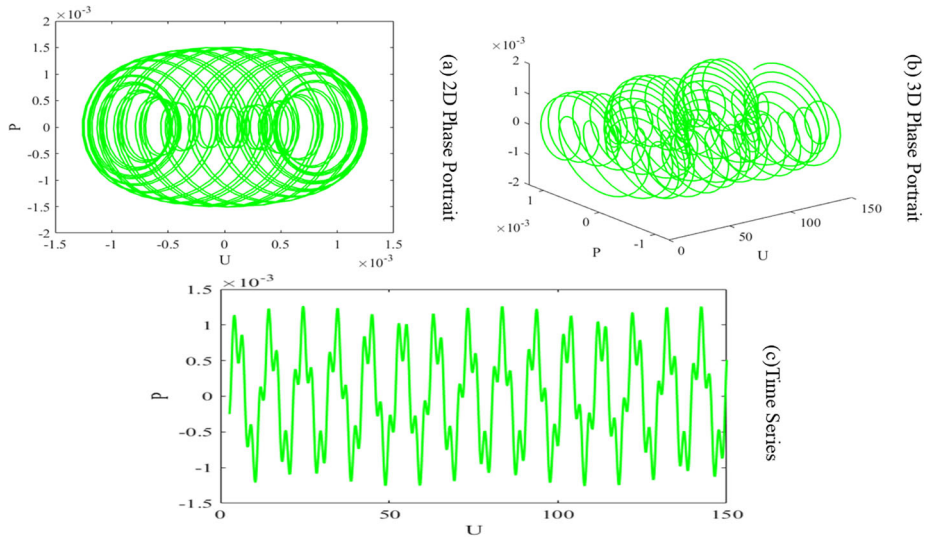

Figure 11 (a) 2D Phase portrait, (b) 3D Phase portrait, (c) Time series of the NLDS Eq. (1), for $f_0 = -0.00049$ and $\omega = 0.64$

5.3 A chaotic arrangement

Now, by considering the properties of the perturbation, we will investigate the chaotic behavior in the dynamical system (37). In this case, the first step in assessing the algorithm is to determine how the relative changes of one parameter affect the resultant effect when all other parameters remain constant. The system (37) with $\omega = 0.64$ and $f_0 = -1.49$ labeled in Fig. 10 shows quasi-periodic movement in time series and phase space. Furthermore, all other factors are held constant at the fixed values as defined in Fig. 10. Substituting the value of $\omega = 0.64$, one can analyze the reaction of the system (37) by changes in f_0. Substituting $f_0 = -0.00049$, the related time series and phase portrait are shown in Fig. 11 (a), (b), and (c). The phase portrait has a possible low stability of the trajectory, and the time series flutters in amplitude. This analysis also illustrates the system's response to parameter variations and how potential chaos may occur.

6 Conclusion

This article uses fractional calculus together with diffusion-reaction equations and integrates the application of GREMM, an effective and novel method for depicting and predicting the behavior of nonlinear dynamic systems. This approach allows one to get a better understanding of the main aspects of such phenomena as solitary wave propagation and bifurcation and thus a better understanding of memory effects and nonlinear dynamics within such systems. Unlike traditional approaches, this method of employing fractional calculus integrates past effects and historical dependencies in systems into the models. This permits the study of different sorts of waves in great detail, as well as nonlinear behavior, chaos, and shifts in stability, including kink waves, pulse waves, and periodically traveling waves. The method looks very encouraging in different fields of science and engineering, including biology, chemistry, and physics, where precise prediction and behavior control are critical. In addition, it improves the prediction, analysis, and control capacity of future nonlinear phenomena for power systems and complicated systems; therefore, this framework is very significant and necessary to study and construct control

strategies for complex systems in the future. This study does not include a formal Lyapunov exponent or Poincaré section analysis, but future work may employ such tools to further quantify the observed chaotic behavior.

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All authors have read and approved the final manuscript.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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