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On the stability and behavior of solutions in mixed differential equations with delays and advances

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In this study, some new results are obtained on asymptotic behavior and stability analysis of the mixed type differential equation

$$x'(t)=\int_{-1}^{0}x(t-r_1(\theta))dv(\theta)+\int_{-1}^{0}x(t+r_2(\theta))d\eta(\theta),$$

where $x(t)\in\mathbb{R}$, $r_1(\theta)$ and $r_2(\theta)$ are real nonnegative continuous functions on $[-1,0]$, and $v(\theta)$ and $\eta(\theta)$ are real valued functions of bounded variation on $[-1,0]$. These results were obtained by using an appropriate real root of the characteristic equation. Moreover, by the use of two appropriate distinct real roots of the corresponding characteristic equation, a new result on the behavior of solutions is established. Five examples are also given to illustrate our results. We also presented the application of the obtained results in the special case of constant coefficients and have given three different cases in one example. The results obtained in this article show that real roots play an important role.

KEYWORDS

asymptotic behavior, characteristic equation, mixed differential equation, stability, zero solution

1 | INTRODUCTION AND PRELIMINARIES

Mixed type functional differential equations are a special class of functional differential equations where the time derivative depends on both past and future values of the variable. The reader is first directed to Rustichini[1] for a brief discussion of the motivations for studying such equations. Later the reader is advised to read articles by Cassell and Hou,[2] Mallet-Paret,[3] Krisztin,[4] Iakovleva and Vanegas,[5] and references therein. Mixed type functional differential equations appear in, for example, optimal control problems,[6] nerve conduction theory,[7] models for economic dynamics,[1,8] and the description of traveling waves in a spatial lattice.[3,9–11]

Consider the differential equation of mixed type

$$x'(t)=\int_{-1}^{0}x(t-r_1(\theta))dv(\theta)+\int_{-1}^{0}x(t+r_2(\theta))d\eta(\theta),\quad t\geq 0, \tag{1.1}$$

where $x(t)\in\mathbb{R}$, $r_1(\theta)$ and $r_2(\theta)$ are real nonnegative continuous functions on $[-1,0]$, and $v(\theta)$ and $\eta(\theta)$ are real valued functions of bounded variation on $[-1,0]$. These integrals are Riemann–Stieltjes integrals. It will be assumed that v and η are not constant on $[-1,0]$.

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Let

$$\|r_1\| = \max\{r_1(\theta):\ -1 \le \theta \le 0\} \text{ and } \|r_2\| = \max\{r_2(\theta):\ -1 \le \theta \le 0\}.$$

Along with the mixed type differential equation (1.1), we specify an *initial condition* of the form

$$x(t) = \phi(t), \quad -\|r_1\| \le t \le \|r_2\|, \tag{1.2}$$

where the initial function ϕ is given a continuous real-valued function on the interval $[-\|r_1\|, \|r_2\|]$ satisfying the *consistency condition*

$$\phi'(0) = \int_{-1}^{0} \phi(-r_1(\theta))dv(\theta) + \int_{-1}^{0} \phi(r_2(\theta))d\eta(\theta).$$

The consistency condition for the initial condition ϕ will be assumed throughout this article.

The initial value problem associated to (1.1–1.2 can be ill-posed. This leads to some questions concerning the admissibility of the function spaces for abstract Cauchy problems of functional differential equations. On this subject, we call the attention of the reader to previous works.[12,13] However, it is possible that this initial value problem can be applied (see previous studies[1,2,5,14–20]). A similar application was made for mixed type difference equations in Yeniçerioglu et al.[21]

Equation (1.1) represents a broad class of linear functional differential equations of a mixed type. The similar equation is considered by Krisztin[4] in some mathematical applications seen in the literature, such as in previous works.[1,2,7,8,22] Also, Malique[23] did his doctorate on the oscillation and numerical field related to Equation (1.1).

When $\eta(\theta)$ is a null function, then Equation (1.1) reduces to the retarded functional differential equation

$$x'(t) = \int_{-1}^{0} x(t - r_1(\theta))dv(\theta).$$

The stability behaviors of this retarded functional differential equation are obtained in previous studies.[24,25] Cooke and Ferreira[24] have achieved the stability of solutions over a cone defined as the region of stability. Yeniçerioglu et al.[25] investigated the asymptotic behavior of the solutions and the stability of the zero solution using a suitable real root for the characteristic equation. Pedro[26–28] established the oscillatory criteria for retarded functional equations. Ford et al.[29] used the numerical method in the same equation. For the retarded functional equations, we naturally extended the theoretical results obtained in Yeniçerioglu et al.[25] to the mixed type differential equation (1.1).

The authors in previous studies[30–32] obtained very interesting results regarding the oscillation behavior of the solutions of Equation (1.1). Ferreira and Pinelas[30,32] have obtained some criteria to guarantee that all solutions of Equation (1.1) are oscillatory. In addition, Pinelas[32] also took into account the existence of nonoscillatory solutions. Ferreira et al.[31] have established some analytical and numerical criteria to ensure that all solutions of Equation (1.1) are oscillatory.

The stability theory of delay differential equations has received a large amount of attention during the last 30 years, as one can see through the textbooks[33–36] and references therein.

By a *solution* of (1.1), we mean a continuous function $x : [-\|r_1\|, +\infty) \to \mathbb{R}$, which is differentiable on $[0, +\infty)$ and satisfies (1.1) for every $t \ge 0$.

If a solution of (1.1) is searched in the form $x(t) = e^{\lambda t}$ for $t \in \mathbb{R}$, the *characteristic equation* of (1.1) will be

$$\lambda = \int_{-1}^{0} e^{-\lambda r_1(\theta)}dv(\theta) + \int_{-1}^{0} e^{\lambda r_2(\theta)}d\eta(\theta). \tag{1.3}$$

Sufficient conditions are obtained for the existence of real roots of the characteristic equation (1.3) in Section 4.

The linear autonomous mixed type differential equation is a special version of Equation (1.1)

$$x'(t) = \sum_{i=1}^{p} a_i x(t - r_i) + \sum_{j=1}^{q} b_j x\big(t + \tau_j\big), \tag{1.4}$$

where a_i and b_j are nonzero real numbers and each r_i and τ_j are positive real numbers ($i = 1,...,p$ and $j = 1,...,q$). Such equations can arise in the study of traveling waves in regions with nonlocal interactions initiated in previous works.[18,19] As is well known, this equation can be obtained from (1.1), under the assumption that $v(\theta)$ and $\eta(\theta)$ are step functions with numbers p and q of jump points. More concretely, it can be obtained from (1.1) with $v(\theta)$ and $\eta(\theta)$ given explicitly, for example, by

$$v(\theta) = \sum_{i=1}^{p} H(\theta - \theta_i) a_i,\ \ \eta(\theta) = \sum_{j=1}^{q} H\big(\theta - \zeta_j\big) b_j,$$

where for $-1 < \theta_1 < ... < \theta_p < 0$ and $-1 < \zeta_1 < ... < \zeta_q < 0$, by H we mean the Heaviside function and $r_1(\theta)$ and $r_2(\theta)$ are continuous and nonnegative functions on $[-1,0]$ such that $r_1(\theta_i) = r_i$ and $r_2\big(\zeta_j\big) = \tau_j$, for $i = 1,...,p$ and $j = 1,...,q$ (see resource for detailed information,[37 Chapter 7.9]).

The reader can look at previous works[14,15] and references therein of the equation

$$x'(t) = ax(t-1) + bx(t+1),$$

which is a special case of Equation 1.4. In these references, existence and uniqueness are obtained for this equation under certain conditions. The numerical approach to this equation can be seen in previous studies.[14,15,38,39]

Mallet-Paret[3] developed a Fredholm theory for mixed-type equations such as Equation 1.4. Later, Mallet-Paret[9] used this Fredholm theory to prove the existence, uniqueness, and other properties of monotone traveling wave solutions of spatially discrete dynamical equations and moreover, Mallet-Paret and Lunel[11] introduced the idea of factorization solutions in the autonomous case by applying the Fredholm theory. Furthermore, exponential dichotomies are created for an equation such as (1.4) (see previous works[10,40] Section 3). Rustichini[1] considered the linear autonomous case, and this author examined the spectrum of the unbounded operator and constructed continuous semigroups on stable, center and unstable subspaces. Similarly, Mallet-Paret and Lunel[10] obtained the decomposition of the state space into stable and unstable subspaces with associated semigroups or evolutionary processes. In the autonomous case, the authors additionally obtained the representations of the semigroups in terms of retarded and advanced equations. In Section 7 of this article, we considered the autonomous case, namely, Equation 1.4. We obtained the stability analysis of the zero solution of Equation 1.4 using an appropriate real root of the characteristic equation.

Iakovlev and Iakovleva[18] used a step derivation method and some conditions in an initial function to guarantee the existence, uniqueness, and smoothness of the solution of the linear inhomogeneous mixed-type differential equation. Also, the authors defined a transformation group on a complete countably normed space of initial functions and examined the spectrum of the infinitesimal generator of this group. Yılmaz and Ismailov[41] examined all boundedly solvable extensions generated by first-order linear inhomogeneous mixed-type functional differential operator expressions in the Hilbert space of vector functions on a finite interval. The authors also obtained the structure of the spectrum of these extensions. In a doctoral thesis in Mbuyongha,,[20 Chapter 2] it was generalized to the rather challenging case of distributed deviating arguments by introducing a recurrence relation and providing the necessary conditions for the existence and uniqueness of solutions of linear mixed-type functional differential equations. In addition, the conditions under which a semigroup theory can be applied to linear mixed type functional differential equations are investigated by providing a function space whose elements satisfy the semigroup requirements. In another PhD, Hupkes[42] provided a center manifold framework for autonomous mixed-type differential equations. Hupkes and Augeraud-Véron[16] reviewed the well-posedness initial value problems for scalar functional algebraic and differential functional equations of mixed type. Similarly, it is shown how the system can be written as a classical Cauchy problem in a suitable Banach space, provided that the initial value problem related to the linear autonomous mixed-type differential equation given in Mármol and Vanegas[19] is well presented. Then the semigroup associated with the ordinary differential equation and its infinitesimal generator are given and some important properties of these operators are proved. In Agarwal et al.,[43 Chapter 5] nonoscillation properties of linear mixed-type differential equations are given and their positive solutions are obtained.

Equation 1.1 for $r_1(\theta) = -a\theta$ $(a > 0)$ and $r_2(\theta) = -b\theta$ $(b > 0)$ is reduced to the class of linear mixed-type functional differential equation

$$x'(t) = \int_{-1}^{0} x(t + a\theta)dv(\theta) + \int_{-1}^{0} x(t - b\theta)d\eta(\theta)$$

or

$$x'(t) = \int_{-a}^{0} x(t + \theta)dv\left(\frac{\theta}{a}\right) - \int_{0}^{b} x(t + \theta)d\eta\left(-\frac{\theta}{b}\right).$$

Then, if we set $\mu(\theta) = v\left(\frac{\theta}{a}\right) = -\eta\left(-\frac{\theta}{b}\right)$, the above equation becomes

$$x'(t) = \int_{-a}^{b} x(t + \theta)d\mu(\theta), \tag{1.5}$$

where $\mu(\theta)$ is real valued function of bounded variation on $[-a, b]$. Note that a special case of (1.5) can be written as in 1.4 by using the step functions as mentioned above. This is one of the most common linear mixed-type functional equations seen in the literature[17] (see also previous works[4,10,23] and the references therein). Equation (1.5) is considered in a deterministic case and an economic application of such an equation is stated. The authors also obtained the necessary conditions for the existence, uniqueness, and stability of a solution of Equation (1.5). In other words, d'Albis et al.[17] have given a precise characterization of the dynamics lying on the stable manifold, using the analytical properties of the model. The authors calculated a threshold value such that all roots whose real part is smaller than this threshold belong to the stable manifold. The authors also proved analytically that, for a given initial condition, the equation that characterizes the dynamics on the stable manifold has a unique solution. However, d'Albis et al.[17] has no information about asymptotic behavior and exponential estimate of solutions. Our aim in this article is to apply a different method than the article in d'Albis et al.[17] for the stability of Equation (1.1). Namely, we obtained the asymptotic behavior of the solutions using an appropriate real root of the characteristic equation (1.3), and then we construct a useful exponential estimate for these solutions and finally provide a stability criterion. Furthermore, by using two appropriate distinct real roots of the corresponding characteristic equation (1.3), we obtained the lower and upper bounds of the solutions. The real roots used in this article will show that they play an important role in determining all results. The techniques used to obtain the results are a combination of the methods used in previous works.[25,44–47]

In reality, there is usually no gain by including arbitrary continuous functions $r_1(\theta)$ and $r_2(\theta)$ in Equation 1.1, but there may be some gain in convenience. That is, a special case of Equation 1.1, Equation 1.4 or 1.5 may arise.

The results given in the next sections (Sections 2–7) are formulated into five theorems (Theorems 2.1, 3.1, 6.1, 7.1, and 7.3), four corollaries (Corollary 2.2, 4.2, 6.2, and 7.2), five lemmas (Lemmas 4.1, 4.3, 4.4, 6.3, and 7.4) and six examples (Examples 5.1, 5.2, 5.3, 5.4, 6.4, and 7.5). Theorem 2.1 in Section 2 creates a basic asymptotic result for the solution of (1.1)–(1.2) and Corollary 2.2 concerns the special case of Theorem 2.1. In Section 3, the estimates of solutions (1.1)–(1.2) and also the stability criteria of the zero solution of (1.1) are determined by Theorem 3.1. Here, Theorems 2.1 and 3.1 are obtained using an appropriate real root λ_0 of the characteristic equation (1.3). In Section 4, Lemma 4.1 establishes sufficient conditions for the characteristic equation (1.3) to have a real root with a suitable property. Using this Lemma, the stability criterion was established in Corollary 4.2. Lemma 4.3 and Lemma 4.4 are related to the real roots of (1.3). In Section 5, examples about stability, instability, and asymptotic stability of the zero solution of (1.1) are given. In Section 6, in deriving Theorem 6.1, two appropriate distinct real roots λ_0 and λ_1 of the characteristic equation (1.3) are used. Lemma 6.3 is an auxiliary result about these two roots of the characteristic equation (1.3). Also, an example was given about Theorem 6.1. In the last section, applications for the special cases of Equation 1.4 are also presented and three different cases (stable, asymptotically stable and unstable) are given in one example.

Throughout this article, by $V(v)$ we denote the total variation function of v, which is defined on the interval $[-1,0]$. Analogously, $V(\eta)$ will stand for the total variation function of η defined on the interval $[-1,0]$. Note that the functions $V(v)$ and $V(\eta)$ are not negative on the interval $[-1,0]$. Furthermore, it must be noted that $V(v)$ and $V(\eta)$ are not

identically zero on the interval $[-1,0]$. In this article, we will estimate the Riemann–Stieltjes integral with the following inequalities:

$$\left|\int_{-1}^{0} h(\theta)dg(\theta)\right| \le \int_{-1}^{0} |h(\theta)|dV(g)(\theta) \le \max_{-1\le\theta\le 0}|h(\theta)|\cdot V(g)(-1,0),$$

where h is continuous function on $[-1,0]$, g is a function of bounded variation on $[-1,0]$, and $V(g)(-1,0)$ is the total variation of g on the interval $[-1,0]$. If g is a monotone function, it becomes

$$\int_{-1}^{0} |h(\theta)|dV(g)(\theta) \le \max_{-1\le\theta\le 0}|h(\theta)|\cdot|g(-1)-g(0)|. \tag{1.6}$$

It is assumed that the reader knows the theory of Riemann–Stieltjes integration and the theory of functions of bounded variation (see previous studies[33,37,48]).

Before we finish this section, we will give three definitions of stability (see, e.g., Hale and Lunel[33]). The zero solution of 1.1 is said to be *stable* if for every $\varepsilon>0$, there exists a number $\ell=\ell(\varepsilon)>0$ such that, for any initial function ϕ with $\|\phi\| = \max_{-\|r_1\|\le t\le\|r_2\|}|\phi(t)| < \ell$, the solution x of (1.1)–(1.2) satisfies

$$|x(t)|<\varepsilon, \quad \textit{for all } t\in[-\|r_1\|,\infty).$$

Otherwise, the zero solution of (1.1) is said to be *unstable*. Moreover, the zero solution of (1.1) is called *asymptotically stable* if it is stable in the above sense, and in addition, there exists a number $\ell_0>0$ such that, for any initial function ϕ with $\|\phi\|<\ell_0$, the solution x of (1.1)–(1.2) satisfies

$$\lim_{t\to\infty} x(t)=0.$$

2 | THE ASYMPTOTIC RESULT

The following theorem gives a fundamental asymptotic criterion for the solutions of (1.1).

Theorem 2.1. *Let λ_0 be a real root of the characteristic equation 1.3 with the property*

$$\mu(\lambda_0)=\int_{-1}^{0} r_1(\theta)e^{-\lambda_0 r_1(\theta)}dV(v)(\theta)+\int_{-1}^{0} r_2(\theta)e^{\lambda_0 r_2(\theta)}dV(\eta)(\theta)<1, \tag{2.1}$$

and set

$$\beta(\lambda_0)=\int_{-1}^{0} r_1(\theta)e^{-\lambda_0 r_1(\theta)}dv(\theta)-\int_{-1}^{0} r_2(\theta)e^{\lambda_0 r_2(\theta)}d\eta(\theta). \tag{2.2}$$

Then, for every $\phi\in C([-\|r_1\|,\|r_2\|],\mathbb{R})$, the solution x of (1.1)–(1.2) satisfies

$$\lim_{t\to\infty}\left[e^{-\lambda_0 t}x(t)\right]=\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)},$$

where

$$L(\lambda_0;\phi)=\phi(0)+\int_{-1}^{0}e^{-\lambda_0 r_1(\theta)}\left(\int_{-r_1(\theta)}^{0}e^{-\lambda_0 s}\phi(s)ds\right)dv(\theta)$$
$$-\int_{-1}^{0}e^{\lambda_0 r_2(\theta)}\left(\int_{0}^{r_2(\theta)}e^{-\lambda_0 s}\phi(s)ds\right)d\eta(\theta). \tag{2.3}$$

Note that property 2.1 guarantees that $1+\beta(\lambda_0)>0$.

Note that we clearly see that $\lambda_0=0$ is a root of (1.3) with the property (2.1) if and only if

$$\int_{-1}^{0}dv(\theta)+\int_{-1}^{0}d\eta(\theta)=0, \tag{2.4}$$

and

$$\int_{-1}^{0}r_1(\theta)dV(v)(\theta)+\int_{-1}^{0}r_2(\theta)dV(\eta)(\theta)<1. \tag{2.5}$$

Thus, an application of Theorem 2.1 with $\lambda_0=0$ leads to the following corollary.

Corollary 2.2. *Let conditions (2.4) and (2.5) be satisfied. Then, for every $\phi\in C([-\|r_1\|,\|r_2\|],\mathbb{R})$, the solution x of (1.1)–(1.2) satisfies*

$$\lim_{t\to\infty}x(t)=\frac{\phi(0)+\int_{-1}^{0}\left(\int_{-r_1(\theta)}^{0}\phi(s)ds\right)dv(\theta)-\int_{-1}^{0}\left(\int_{0}^{r_2(\theta)}\phi(s)ds\right)d\eta(\theta)}{1+\int_{-1}^{0}r_1(\theta)dv(\theta)-\int_{-1}^{0}r_2(\theta)d\eta(\theta)}.$$

Note that the assumption of (2.5) provides that

$$1+\int_{-1}^{0}r_1(\theta)dv(\theta)-\int_{-1}^{0}r_2(\theta)d\eta(\theta)>0.$$

Proof of Theorem 2.1. Property 2.1 guarantees that $1+\beta(\lambda_0)>0$. Indeed, from (2.2), we have

$$|\beta(\lambda_0)|=\left|\int_{-1}^{0}r_1(\theta)e^{-\lambda_0 r_1(\theta)}dv(\theta)-\int_{-1}^{0}r_2(\theta)e^{\lambda_0 r_2(\theta)}d\eta(\theta)\right|,$$
$$\leq\int_{-1}^{0}r_1(\theta)e^{-\lambda_0 r_1(\theta)}dV(v)(\theta)+\int_{-1}^{0}r_2(\theta)e^{\lambda_0 r_2(\theta)}dV(\eta)(\theta),$$

and using property 2.1, we get $|\beta(\lambda_0)|\leq\mu(\lambda_0)$. So, it holds $|\beta(\lambda_0)|<1$. This implies that $1+\beta(\lambda_0)>0$.

Consider any initial function ϕ in $C([-\|r_1\|,\|r_2\|],\mathbb{R})$ and let x be the solution of (1.1)–(1.2). Define

$$y(t)=e^{-\lambda_0 t}x(t),\text{for } t\in[-\|r_1\|,\infty).$$

Then, it is obtained for every $t\geq 0$

$$y'(t)+\lambda_0 y(t)=\int_{-1}^{0}e^{-\lambda_0 r_1(\theta)}y(t-r_1(\theta))dV(\theta)+\int_{-1}^{0}e^{\lambda_0 r_2(\theta)}y(t+r_2(\theta))d\eta(\theta). \tag{2.6}$$

Moreover, the initial condition (1.2) can be equivalently written as

$$y(t)=e^{-\lambda_0 t}\phi(t)\ \text{ for }\ t\in[-\|r_1\|,\|r_2\|]. \tag{2.7}$$

Using the fact that λ_0 is a real root of (1.3) and that (2.7) is taken into account, from (2.6) we obtain

$$\begin{aligned}
&y(t)-y(0)+\lambda_0\int_0^t y(s)ds\\
&=\int_0^t\left(\int_{-1}^{0}e^{-\lambda_0 r_1(\theta)}y(s-r_1(\theta))dv(\theta)+\int_{-1}^{0}e^{\lambda_0 r_2(\theta)}y(s+r_2(\theta))d\eta(\theta)\right)ds,\\
&y(t)=\phi(0)-\lambda_0\int_0^t y(s)ds+\int_{-1}^{0}e^{-\lambda_0 r_1(\theta)}\left(\int_0^t y(s-r_1(\theta))ds\right)dv(\theta)\\
&\qquad+\int_{-1}^{0}e^{\lambda_0 r_2(\theta)}\left(\int_0^t y(s+r_2(\theta))ds\right)d\eta(\theta),\\
&y(t)=\phi(0)-\left(\int_{-1}^{0}e^{-\lambda_0 r_1(\theta)}dv(\theta)+\int_{-1}^{0}e^{\lambda_0 r_2(\theta)}d\eta(\theta)\right)\int_0^t y(s)ds\\
&\qquad+\int_{-1}^{0}e^{-\lambda_0 r_1(\theta)}\left(\int_{-r_1(\theta)}^{t-r_1(\theta)}y(s)ds\right)dv(\theta)+\int_{-1}^{0}e^{\lambda_0 r_2(\theta)}\left(\int_{r_2(\theta)}^{t+r_2(\theta)}y(s)ds\right)d\eta(\theta),\\
&y(t)=\phi(0)-\int_{-1}^{0}e^{-\lambda_0 r_1(\theta)}\left(\int_0^t y(s)ds\right)dv(\theta)-\int_{-1}^{0}e^{\lambda_0 r_2(\theta)}\left(\int_0^t y(s)ds\right)d\eta(\theta)\\
&\qquad+\int_{-1}^{0}e^{-\lambda_0 r_1(\theta)}\left(\int_{-r_1(\theta)}^{0}e^{-\lambda_0 s}\phi(s)ds\right)dv(\theta)+\int_{-1}^{0}e^{-\lambda_0 r_1(\theta)}\left(\int_0^{t-r_1(\theta)}y(s)ds\right)dv(\theta)\\
&\qquad+\int_{-1}^{0}e^{\lambda_0 r_2(\theta)}\left(\int_{r_2(\theta)}^{t}y(s)ds+\int_t^{t+r_2(\theta)}y(s)ds\right)d\eta(\theta),\\
&y(t)=L(\lambda_0;\phi)-\int_{-1}^{0}e^{-\lambda_0 r_1(\theta)}\left(\int_{t-r_1(\theta)}^{t}y(s)ds\right)dv(\theta)+\int_{-1}^{0}e^{\lambda_0 r_2(\theta)}\left(\int_t^{t+r_2(\theta)}y(s)ds\right)d\eta(\theta),
\end{aligned} \tag{2.8}$$

where $L(\lambda_0;\phi)$ is defined as in Theorem 2.1. Next, we set for $t\geq -\|r_1\|$

$$z(t) = y(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}.$$

Then Equation 2.8 is reduced to the following equation for $t \geq 0$

$$z(t) = -\int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} \left(\int_{t-r_1(\theta)}^{t} z(s)ds \right) dv(\theta) + \int_{-1}^{0} e^{\lambda_0 r_2(\theta)} \left(\int_{t}^{t+r_2(\theta)} z(s)ds \right) d\eta(\theta). \tag{2.9}$$

Moreover, (2.7) is written as for $t \in [-\|r_1\|, \|r_2\|]$

$$z(t) = e^{-\lambda_0 t}\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}. \tag{2.10}$$

By the definition of y and z, what we have to prove is that

$$\lim_{t\to\infty} z(t) = 0. \tag{2.11}$$

Put

$$M(\lambda_0;\phi) = \max_{t\in[-\|r_1\|,\|r_2\|]} \left| e^{-\lambda_0 t}\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} \right|. \tag{2.12}$$

Then, in view of (2.10), we have

$$|z(t)| \leq M(\lambda_0;\phi) \text{ for } -\|r_1\| \leq t \leq 0. \tag{2.13}$$

We will show that $M(\lambda_0;\phi)$ is a bound of z on the whole interval $[-\|r_1\|, \infty)$, namely, that

$$|z(t)| \leq M(\lambda_0;\phi) \text{ for all } t \geq -\|r_1\|. \tag{2.14}$$

To this end, let's consider an arbitrary number $\varepsilon > 0$. It is claimed that

$$|z(t)| < M(\lambda_0;\phi) + \varepsilon \text{ for every } t \geq -\|r_1\|. \tag{2.15}$$

Otherwise, because of (2.13), there exists a point $t_0 > 0$ such that

$$|z(t)| < M(\lambda_0;\phi) + \varepsilon \text{ for } t \in [-\|r_1\|, t_0) \cup (t_0, t_0 + \|r_2\|] \text{ and } |z(t_0)| = M(\lambda_0;\phi) + \varepsilon.$$

Then, using (2.1), from (2.9) we obtain

$$\begin{aligned}
\mathrm{M}(\lambda_0;\phi) + \varepsilon = |\mathrm{z}(\mathrm{t}_0)| &= \left| -\int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} \left(\int_{t_0-r_1(\theta)}^{t_0} z(s)ds \right) dv(\theta) + \int_{-1}^{0} e^{\lambda_0 r_2(\theta)} \left(\int_{t_0}^{t_0+r_2(\theta)} z(s)ds \right) d\eta(\theta) \right| \\
&\leq \int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} \left(\int_{t_0-r_1(\theta)}^{t_0} |z(s)|ds \right) dV(v)(\theta) + \int_{-1}^{0} e^{\lambda_0 r_2(\theta)} \left(\int_{t_0}^{t_0+r_2(\theta)} |z(s)|ds \right) dV(\eta)(\theta) \\
&\leq [M(\lambda_0;\phi) + \varepsilon] \left(\int_{-1}^{0} r_1(\theta) e^{-\lambda_0 r_1(\theta)} dV(v)(\theta) + \int_{-1}^{0} r_2(\theta) e^{\lambda_0 r_2(\theta)} dV(\eta)(\theta) \right) \\
&= [M(\lambda_0;\phi) + \varepsilon]\mu(\lambda_0) < [M(\lambda_0;\phi) + \varepsilon],
\end{aligned}$$

which in view of (2.1) leads to a contradiction. Therefore, our claim is true; that is, (2.15) is provided. It has been thus proved that (2.15) is satisfied for all numbers $\varepsilon > 0$. Therefore, (2.14) is provided. By virtue of (2.14), from (2.9) we derive for $t \geq 0$,

$$|z(t)| \leq \int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} \left(\int_{t-r_1(\theta)}^{t} |z(s)|ds \right) dV(v)(\theta) + \int_{-1}^{0} e^{\lambda_0 r_2(\theta)} \left(\int_{t}^{t+r_2(\theta)} |z(s)|ds \right) dV(\eta)(\theta) \leq M(\lambda_0;\phi) \left(\int_{-1}^{0} r_1(\theta) e^{-\lambda_0 r_1(\theta)} dV(v)(\theta) + \int_{-1}^{0} r_2(\theta) e^{\lambda_0 r_2(\theta)} dV(\eta)(\theta) \right).$$

Consequently, by the definition of $\mu(\lambda_0)$, we have

$$|z(t)| \leq \mu(\lambda_0) M(\lambda_0;\phi) \text{ for every } t \geq 0. \tag{2.16}$$

Using (2.9) and taking into consideration the definition of $\mu(\lambda_0)$ as well as (2.14) and (2.16), it can be shown, by an easy induction, that z satisfies

$$|z(t)| \leq [\mu(\lambda_0)]^n M(\lambda_0;\phi) \text{ for all } t \geq n\|r_1\| - \|r_1\| \ (n = 0,1,2,\ldots). \tag{2.17}$$

Because of (2.1), we have $\lim_{n\to\infty} [\mu(\lambda_0)]^n = 0$. Thus, from (2.17), it follows that $\lim_{t\to\infty} z(t) = 0$, that is, (2.11) holds. The proof of Theorem 2.1 is complete.

3 | ESTIMATION OF SOLUTIONS AND STABILITY CRITERIA

The following theorem establishes an estimate for the solutions of mixed type functional differential equations (1.1) and also a stability criterion for the zero solution of (1.1).

Theorem 3.1. *Let λ_0 be a real root of (1.3) with the property 2.1. Consider $\mu(\lambda_0)$ and $\beta(\lambda_0)$ defined by (2.1) and (2.2), respectively. Then, for every $\phi \in C([-\|r_1\|, \|r_2\|], \mathbb{R})$, the solution x of (1.1)–(1.2) satisfies*

$$|x(t)| \leq \left[\frac{(1+\mu(\lambda_0))^2}{1+\beta(\lambda_0)} + \mu(\lambda_0) \right] N(\lambda_0;\phi) e^{\lambda_0 t} \quad \text{for all } t \geq 0, \tag{3.1}$$

where

$$N(\lambda_0;\phi) = \max_{-\|r_1\| \leq t \leq \|r_2\|} \left| e^{-\lambda_0 t} \phi(t) \right|. \tag{3.2}$$

Moreover, the zero solution of (1.1) is stable if $\lambda_0 = 0$, asymptotically stable if $\lambda_0 < 0$, and unstable if $\lambda_0 > 0$.

Proof. Consider any function ϕ in $C([-\|r_1\|, \|r_2\|], \mathbb{R})$ and let x be the solution of (1.1)–(1.2). Let y and z be defined as in the proof of Theorem 2.1, that is,

$$y(t) = e^{-\lambda_0 t} x(t) \text{ and } z(t) = y(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} \quad \text{for } t \geq -\|r_1\|,$$

where $L(\lambda_0;\phi)$ is defined by (2.3). Moreover, let $M(\lambda_0;\phi)$ be defined by (2.12). Then, as in the proof of Theorem 2.1, it can be shown that z satisfies (2.16). By the definition of z, from inequality (2.16), it follows that for $t \geq 0$

$$|y(t)| \le \frac{|L(\lambda_0;\phi)|}{1+\beta(\lambda_0)} + \mu(\lambda_0)M(\lambda_0;\phi). \tag{3.3}$$

By using (2.1) and (3.2), from (2.3) we obtain

$$\begin{aligned}|L(\lambda_0;\phi)| &\le |\phi(0)| + \int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} \left(\int_{-r_1(\theta)}^{0} \left| e^{-\lambda_0 s}\phi(s) \right| ds \right) dV(v)(\theta) \\ &+ \int_{-1}^{0} e^{\lambda_0 r_2(\theta)} \left(\int_{0}^{r_2(\theta)} \left| e^{-\lambda_0 s}\phi(s) \right| ds \right) dV(\eta)(\theta) \le \left(1 + \int_{-1}^{0} r_1(\theta) e^{-\lambda_0 r_1(\theta)} dV(v)(\theta) + \int_{-1}^{0} r_2(\theta) e^{\lambda_0 r_2(\theta)} dV(\eta)(\theta) \right) N(\lambda_0;\phi) \\ &= (1+\mu(\lambda_0))N(\lambda_0;\phi).\end{aligned}$$

On the other hand, from (2.12) we get

$$\begin{aligned}M(\lambda_0;\phi) &\le \max_{t\in[-\|r_1\|,\|r_2\|]} \left| e^{-\lambda_0 t}\phi(t) \right| + \frac{|L(\lambda_0;\phi)|}{1+\beta(\lambda_0)} = N(\lambda_0;\phi) + \frac{|L(\lambda_0;\phi)|}{1+\beta(\lambda_0)} \\ &\le N(\lambda_0;\phi) + \frac{(1+\mu(\lambda_0))N(\lambda_0;\phi)}{1+\beta(\lambda_0)} = \left(1 + \frac{1+\mu(\lambda_0)}{1+\beta(\lambda_0)} \right) N(\lambda_0;\phi).\end{aligned}$$

So, (3.3) gives

$$|y(t)| \le \frac{(1+\mu(\lambda_0))N(\lambda_0;\phi)}{1+\beta(\lambda_0)} + \mu(\lambda_0)\left(1 + \frac{1+\mu(\lambda_0)}{1+\beta(\lambda_0)} \right) N(\lambda_0;\phi) = \left[\frac{(1+\mu(\lambda_0))^2}{1+\beta(\lambda_0)} + \mu(\lambda_0) \right] N(\lambda_0;\phi).$$

As a result, according to the definition of y, we obtain

$$|x(t)| \le \left[\frac{(1+\mu(\lambda_0))^2}{1+\beta(\lambda_0)} + \mu(\lambda_0) \right] N(\lambda_0;\phi) e^{\lambda_0 t} \quad \text{for all } t \ge 0.$$

So, we have completed the proof of the first part of this theorem. Now let's prove the second part (stability criterion) of the theorem.

Let's assume that $\lambda_0 \le 0$. Let $\phi \in C([-\|r_1\|, \|r_2\|], \mathbb{R})$ be any initial function and let x be the solution of (1.1)–(1.2). Then (3.1) holds and hence

$$|x(t)| \le \left[\frac{(1+\mu(\lambda_0))^2}{1+\beta(\lambda_0)} + \mu(\lambda_0) \right] N(\lambda_0;\phi) \text{ for all } t \ge 0.$$

Since $\frac{(1+\mu(\lambda_0))^2}{1+\beta(\lambda_0)} > 1$, it follows that

$$|x(t)| \le \left[\frac{(1+\mu(\lambda_0))^2}{1+\beta(\lambda_0)} + \mu(\lambda_0) \right] N(\lambda_0;\phi) \text{ for all } t \ge -\|r_1\|.$$

From the last inequality, we can verify that the zero solution of (1.1) is stable. Moreover, if $\lambda_0 < 0$, then (3.1) guarantees that

$$\lim_{t\to\infty} x(t) = 0.$$

Thus, for $\lambda_0 < 0$, the zero solution of (1.1) is asymptotically stable.

As a result, we suppose that $\lambda_0 > 0$ and it will be proved that the zero solution of (1.1) is unstable. Otherwise, assume that the zero solution of (1.1) is stable. Then it can be chosen a number $\delta > 0$ such that, for each $\phi \in C([-\|r_1\|, \|r_2\|], \mathbb{R})$ with $\|\phi\| < \delta$, the solution x of (1.1)–(1.2) satisfies

$$|x(t)| < 1 \text{ for } t \geq -\|r_1\|. \tag{3.4}$$

Set

$$\phi_0(t) = e^{\lambda_0 t} \text{ for } t \in [-\|r_1\|, \|r_2\|].$$

We see that $\phi_0 \in C([-\|r_1\|, \|r_2\|], \mathbb{R})$ and, from (2.3), we have

$$\begin{aligned} L(\lambda_0;\phi_0) &\equiv \phi_0(0) + \int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} \left(\int_{-r_1(\theta)}^{0} e^{-\lambda_0 s} \phi_0(s) ds \right) dv(\theta) - \int_{-1}^{0} e^{\lambda_0 r_2(\theta)} \left(\int_{0}^{r_2(\theta)} e^{-\lambda_0 s} \phi_0(s) ds \right) d\eta(\theta) \\ &= 1 + \int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} \left(\int_{-r_1(\theta)}^{0} e^{-\lambda_0 s} e^{\lambda_0 s} ds \right) dv(\theta) - \int_{-1}^{0} e^{\lambda_0 r_2(\theta)} \left(\int_{0}^{r_2(\theta)} e^{-\lambda_0 s} e^{\lambda_0 s} ds \right) d\eta(\theta) \\ &= 1 + \int_{-1}^{0} r_1(\theta) e^{-\lambda_0 r_1(\theta)} dv(\theta) - \int_{-1}^{0} r_2(\theta) e^{\lambda_0 r_2(\theta)} d\eta(\theta) \\ &= 1 + \beta(\lambda_0) > 0. \end{aligned} \tag{3.5}$$

Next, we consider a number δ_0 with $0 < \delta_0 < \delta$ and we put

$$\phi = \frac{\delta_0}{\|\phi_0\|} \phi_0.$$

Clearly, ϕ belongs to $C([-\|r_1\|, \|r_2\|], \mathbb{R})$ and $\|\phi\| = \delta_0 < \delta$. Therefore, for this initial function, the solution x of (1.1)–(1.2) satisfies (3.4). Moreover, by using Theorem 2.1 and taking into consideration (3.5) as well as the linearity of the operator $L(\lambda_0;.)$, the following is obtained

$$\lim_{t\to\infty} \left[e^{-\lambda_0 t} x(t) \right] = \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} = \frac{(\delta_0/\|\phi_0\|) L(\lambda_0;\phi_0)}{1+\beta(\lambda_0)} = \frac{\delta_0}{\|\phi_0\|} > 0.$$

But since $\lambda_0 > 0$, from (3.4) it follows that

$$\lim_{t\to\infty} \left[e^{-\lambda_0 t} x(t) \right] = 0.$$

We reached a contradiction here and the second part of the theorem was proved. Thus, the proof of Theorem 3.1 is now completed.

4 | ADDITIONAL LEMMAS

In this section, we give some conditions about the real roots of the characteristic equation (1.3). Here, we will give three lemmas about the real roots of (1.3).

Lemma 4.1. Assume that

$$\int_{-1}^{0} e^{\frac{r_1(\theta)}{r}} dv(\theta) + \int_{-1}^{0} e^{-\frac{r_2(\theta)}{r}} d\eta(\theta) > -\frac{1}{r}, \tag{4.1}$$

$$\int_{-1}^{0} e^{-\frac{r_1(\theta)}{r}} dv(\theta) + \int_{-1}^{0} e^{\frac{r_2(\theta)}{r}} d\eta(\theta) < \frac{1}{r}, \tag{4.2}$$

and

$$\int_{-1}^{0} r_1(\theta) e^{\frac{r_1(\theta)}{r}} dV(v)(\theta) + \int_{-1}^{0} r_2(\theta) e^{\frac{r_2(\theta)}{r}} dV(\eta)(\theta) \leq 1, \tag{4.3}$$

where

$$r = max\{ \|r_1\|, \|r_2\| \}. \tag{4.4}$$

Then, in the interval $\left(-\frac{1}{r}, \frac{1}{r}\right)$, the characteristic equation (1.3) has a unique root λ_0, and this root satisfies the property 2.1.

Proof. Define

$$F(\lambda) = \lambda - \int_{-1}^{0} e^{-\lambda r_1(\theta)} dv(\theta) - \int_{-1}^{0} e^{\lambda r_2(\theta)} d\eta(\theta) \quad \text{for } \lambda \in \left[-\frac{1}{r}, \frac{1}{r}\right].$$

We have

$$F\left(-\frac{1}{r}\right) = -\frac{1}{r} - \int_{-1}^{0} e^{\frac{r_1(\theta)}{r}} dv(\theta) - \int_{-1}^{0} e^{-\frac{r_2(\theta)}{r}} d\eta(\theta.)$$

and so, by (4.1), we get $F\left(-\frac{1}{r}\right) < 0$. Moreover,

$$F\left(\frac{1}{r}\right) = \frac{1}{r} - \int_{-1}^{0} e^{-\frac{r_1(\theta)}{r}} dv(\theta) - \int_{-1}^{0} e^{\frac{r_2(\theta)}{r}} d\eta(\theta),$$

and from (4.2), it follows that $F\left(\frac{1}{r}\right) > 0$. Furthermore, by taking into account (4.3), for $\lambda \in \left(-\frac{1}{r}, \frac{1}{r}\right)$, we obtain

$$\begin{aligned}
F'(\lambda) &= 1 + \int_{-1}^{0} r_1(\theta) e^{-\lambda r_1(\theta)} dv(\theta) - \int_{-1}^{0} r_2(\theta) e^{\lambda r_2(\theta)} d\eta(\theta) \\
&\geq 1 - \left| \int_{-1}^{0} r_1(\theta) e^{-\lambda r_1(\theta)} dv(\theta) \right| - \left| \int_{-1}^{0} r_2(\theta) e^{\lambda r_2(\theta)} d\eta(\theta) \right| \\
&\geq 1 - \int_{-1}^{0} r_1(\theta) e^{-\lambda r_1(\theta)} dV(v)(\theta) - \int_{-1}^{0} r_2(\theta) e^{\lambda r_2(\theta)} dV(\eta)(\theta) \\
&> 1 - \int_{-1}^{0} r_1(\theta) e^{\frac{r_1(\theta)}{r}} dV(v)(\theta) - \int_{-1}^{0} r_2(\theta) e^{\frac{r_2(\theta)}{r}} dV(\eta)(\theta) \geq 0.
\end{aligned}$$

Therefore, F is strictly increasing on the interval $\left(-\frac{1}{r}, \frac{1}{r}\right)$. So, in the interval $\left(-\frac{1}{r}, \frac{1}{r}\right)$, the equation $F(\lambda) = 0$ (which coincides with (1.3)) has a unique root λ_0. This root satisfies (2.1). Indeed, by using again (4.3), we have

$$\begin{aligned}
&\int_{-1}^{0} r_1(\theta) e^{-\lambda_0 r_1(\theta)} dV(v)(\theta) + \int_{-1}^{0} r_2(\theta) e^{\lambda_0 r_2(\theta)} dV(\eta)(\theta) \\
&< \int_{-1}^{0} r_1(\theta) e^{\frac{r_1(\theta)}{r}} dV(v)(\theta) + \int_{-1}^{0} r_2(\theta) e^{\frac{r_2(\theta)}{r}} dV(\eta)(\theta) \leq 1.
\end{aligned}$$

This completes the proof.

We can use Lemma 4.1 and the stability criterion contained in Theorem 3.1 to derive the following result.

Corollary 4.2. *Assume that (4.1)–(4.3) are satisfied. Then the zero solution of (1.1) is asymptotically stable if*

$$v(-1) + \eta(-1) > v(0) + \eta(0),$$

and it is unstable if

$$v(-1) + \eta(-1) < v(0) + \eta(0).$$

Proof. Lemma 4.1 guarantees that, in the interval $\left(-\frac{1}{r}, \frac{1}{r}\right)$, the characteristic equation (1.3) has a unique root λ_0; this root satisfies the property 2.1. Let F be defined as in the proof of Lemma 4.1. For this function, as in the proof of Lemma 4.1, we have

$$F\left(-\frac{1}{r}\right) < 0 \quad \text{and} \quad F\left(\frac{1}{r}\right) > 0.$$

Clearly, λ_0 is negative if $F(0) > 0$, and λ_0 is positive if $F(0) < 0$. On the other hand,

$$F(0) = -\int_{-1}^{0} dv(\theta) - \int_{-1}^{0} d\eta(\theta) = -[v(0) - v(-1) + \eta(0) - \eta(-1)].$$

So, $\lambda_0 < 0$ if $v(-1) + \eta(-1) > v(0) + \eta(0)$, and $\lambda_0 > 0$ if $v(-1) + \eta(-1) < v(0) + \eta(0)$. The proof of Corollary 4.2 is now complete.

Lemma 4.3. Suppose that v and η are decreasing on $[-1,0]$, and r be defined by (4.4). Assume that statement (4.1) is true. Then,

I. In the interval $[0,\infty)$, the characteristic Equation 1.3 has no roots,
II. $\lambda=-\frac{1}{r}$ is not a root of the characteristic Equation 1.3,
III. (i) In the interval $\left(-\frac{1}{r},0\right)$, the characteristic equation 1.3 has a unique root and (ii) in the interval $\left(-\infty,\ -\frac{1}{r}\right)$, the characteristic equation (1.3) has a unique root.

Proof. (I) We first observe that, if λ_0 is a real root of the characteristic equation (1.3), then

$$\lambda_0=\int_{-1}^{0}e^{-\lambda_0 r_1(\theta)}dv(\theta)+\int_{-1}^{0}e^{\lambda_0 r_2(\theta)}d\eta(\theta)<0,$$

and so $\lambda_0<0$. This shows Part (I).

(II) We introduce the function F defined by

$$F(\lambda)=\lambda-\int_{-1}^{0}e^{-\lambda r_1(\theta)}dv(\theta)-\int_{-1}^{0}e^{\lambda r_2(\theta)}d\eta(\theta)\quad\text{for }\lambda\in\mathbb{R}. \tag{4.5}$$

For this purpose, let us assume that (4.1) holds. Assumption 4.1 means that

$$F\left(-\frac{1}{r}\right)=-\frac{1}{r}-\int_{-1}^{0}e^{\frac{r_1(\theta)}{r}}dv(\theta)-\int_{-1}^{0}e^{-\frac{r_2(\theta)}{r}}d\eta(\theta)<0. \tag{4.6}$$

This, in particular, implies that $\lambda=-\frac{1}{r}$ is not a root of the characteristic equation (1.3).

(III) From (4.5), we immediately obtain

$$F'(\lambda)=1+\int_{-1}^{0}r_1(\theta)e^{-\lambda r_1(\theta)}dv(\theta)-\int_{-1}^{0}r_2(\theta)e^{\lambda r_2(\theta)}d\eta(\theta)\quad\text{for }\lambda\in\mathbb{R}. \tag{4.7}$$

Furthermore, we have

$$F''(\lambda)=-\int_{-1}^{0}r_1^2(\theta)e^{-\lambda r_1(\theta)}dv(\theta)-\int_{-1}^{0}r_2^2(\theta)e^{\lambda r_2(\theta)}d\eta(\theta)\quad\text{for }\lambda\in\mathbb{R}. \tag{4.8}$$

So, using the hypothesis that v and η are decreasing, we conclude that

$$F''(\lambda)>0\quad\text{for all }\lambda\in\mathbb{R}, \tag{4.9}$$

and consequently F is convex for $\lambda\in\mathbb{R}$. Next, we observe that

$$F(0)=-\int_{-1}^{0}dv(\theta)-\int_{-1}^{0}d\eta(\theta)>0. \tag{4.10}$$

Furthermore, it is not difficult to show that

$$F(-\infty)=\infty. \tag{4.11}$$

Therefore, from (4.6), (4.9), and (4.10), it follows that in the interval $\left(-\frac{1}{r},0\right)$, 1.3 has a unique root. Moreover, from (4.6), (4.9), and (4.11) guarantee that in the interval $\left(-\infty,\ -\frac{1}{r}\right)$, (1.3) has also a unique root and so Part (III) has been proved.

The proof of the lemma is complete.

Lemma 4.4. Suppose that v and η are increasing on $[-1,0]$, and r be defined by (4.4). Assume that statement (4.2) is true. Then,

I. In the interval $(-\infty,0]$, the characteristic equation (1.3) has no roots,
II. $\lambda=\frac{1}{r}$ is not a root of the characteristic equation (1.3),
III. (i) In the interval $\left(0,\ \frac{1}{r}\right)$, the characteristic equation (1.3) has a unique root and (ii) in the interval $\left(\frac{1}{r},\infty\right)$, the characteristic equation (1.3) has a unique root.

Proof. Consider the real-valued function defined by (4.5). As in the proof of Lemma 4.3, we see that

$$F\left(\frac{1}{r}\right)>0,\quad F(0)<0,\quad F(\infty)=-\infty\quad\text{and}\quad F''(\lambda)<0\ \text{ for all } \lambda\in\mathbb{R}.$$

Thus, the proof will be omitted since it is similar to that of Lemma 4.3.

5 | EXAMPLES

In the following examples, we will apply the stability criteria of the Theorem 3.1. For this, in each example, first an appropriate root of the characteristic equation (1.3) is found. Later the condition (2.1) (i.e., $|\beta(\lambda_0)|\le\mu(\lambda_0)<1$) for the appropriate root λ_0 is applied and finally, the stability of the zero solution of (1.1) is determined. Note that we are using the property 1.6 to show that the condition (2.1) is satisfied.

Example 5.1. Consider Equation 1.1 for $r_1(\theta)=-\theta$, $r_2(\theta)=\theta+1$, $v(\theta)=-\frac{\theta}{4}$ and $\eta(\theta)=\frac{\theta}{4}+1$. In this example, we apply the characteristic equation (1.3). That is,

$$\begin{aligned}\lambda&=\int_{-1}^{0}e^{-\lambda(-\theta)}d\left(-\frac{\theta}{4}\right)+\int_{-1}^{0}e^{\lambda(\theta+1)}d\left(\frac{\theta}{4}+1\right)\\&=-\frac{1}{4}\int_{-1}^{0}e^{\lambda\theta}d\theta+\frac{1}{4}\int_{-1}^{0}e^{\lambda(\theta+1)}d\theta=\frac{1}{4\lambda}\left(e^{\lambda}+e^{-\lambda}-2\right).\end{aligned} \tag{5.1}$$

The characteristic function of (5.1) is

$$F_1(\lambda)=\lambda-\frac{1}{4\lambda}\left(e^{\lambda}+e^{-\lambda}-2\right),$$

and its graph is given in Figure 1.

The graph of the function F_1 shows that three roots of (5.1) are passed. We see that $\lambda=0$ and $\lambda\cong\mp4.36$ are roots of (5.1). Let $\lambda_0=0$. Since v is decreasing on $[-1,0]$ and η is increasing on $[-1,0]$, from (2.1) we get



FIGURE 1 Locations of the roots of 5.1

$$\begin{aligned}\mu(\lambda_0)=\mu(0)&=\int_{-1}^{0}(-\theta)dV\left(-\frac{\theta}{4}\right)+\int_{-1}^{0}(\theta+1)dV\left(\frac{\theta}{4}+1\right)\\&\le\left[\max_{-1\le\theta\le 0}(-\theta)\right]V\left(-\frac{\theta}{4}\right)(-1,0)+\left[\max_{-1\le\theta\le 0}(\theta+1)\right]V\left(\frac{\theta}{4}+1\right)(-1,0)\\&=1\cdot\frac{1}{4}+1\cdot\frac{1}{4}=\frac{1}{2}<1.\end{aligned}$$

Therefore, for $\lambda_0=0$, the condition of Theorem 3.1 is provided. So, the zero solution of (1.1) is stable.
Note that the condition (2.1) of Theorem 3.1 will not be satisfied for the roots $\lambda_0\cong\mp 4.36$. Using (2.2), we obtain

$$\begin{aligned}|\beta(\mp 4.36)|&=\left|\int_{-1}^{0}(-\theta)e^{\mp 4.36\,\theta}d\left(-\frac{\theta}{4}\right)-\int_{-1}^{0}(\theta+1)e^{\mp 4.36\,(\theta+1)}d\left(\frac{\theta}{4}+1\right)\right|\\&=\frac{1}{4}\left|\int_{-1}^{0}\theta e^{\mp 4.36\,\theta}d\theta-\int_{-1}^{0}(\theta+1)e^{\mp 4.36\,(\theta+1)}d\theta\right|=\frac{1}{4}|-13.934|\cong 3.48\le\mu(\mp 4.36),\end{aligned}$$

and thus $\mu(\mp 4.36)>1$ is obtained.

Example 5.2. Consider Equation 1.1 for $r_1(\theta)=\frac{\theta+1}{2}$, $r_2(\theta)=-\frac{\theta}{4}$, $v(\theta)=\frac{(\theta+1)^2}{4}$, and $\eta(\theta)=-\frac{3}{2}\theta$. Then, the characteristic equation (1.3) is

$$\begin{aligned}\lambda&=\int_{-1}^{0}e^{-\lambda\left(\frac{\theta+1}{2}\right)}d\left(\frac{(\theta+1)^2}{4}\right)+\int_{-1}^{0}e^{\lambda\left(-\frac{\theta}{4}\right)}d\left(-\frac{3}{2}\theta\right)\\&=\frac{1}{2}\int_{-1}^{0}\left[e^{-\lambda\left(\frac{\theta+1}{2}\right)}(\theta+1)-3e^{-\frac{\lambda\theta}{4}}\right]d\theta=\frac{1}{\lambda}\left[\frac{2}{\lambda}\left(1-e^{-\frac{\lambda}{2}}\right)-e^{-\frac{\lambda}{2}}-6\left(e^{\frac{\lambda}{4}}-1\right)\right].\end{aligned}\tag{5.2}$$

The characteristic function of (5.2) is

$$F_2(\lambda)=\lambda-\frac{1}{\lambda}\left[\frac{2}{\lambda}\left(1-e^{-\frac{\lambda}{2}}\right)-e^{-\frac{\lambda}{2}}-6\left(e^{\frac{\lambda}{4}}-1\right)\right],$$

and its graph is given in Figure 2.

In the graph of the function F_2, it shows that (5.2) has only one root and we see that $\lambda \cong -0.98$ is the unique root of (5.2). Then, for $\lambda_0 = -0.98$, the condition of Theorem 3.1 is satisfied. That is, since ν is increasing on $[-1,0]$ and η is decreasing on $[-1,0]$,

$$\begin{aligned}\mu(\lambda_0) = \mu(-0.98) &= \int_{-1}^{0} e^{0.98\left(\frac{\theta+1}{2}\right)}\left(\frac{\theta+1}{2}\right)dV\left(\frac{(\theta+1)^2}{4}\right) + \int_{-1}^{0} e^{0.98\left(\frac{\theta}{4}\right)}\left(-\frac{\theta}{4}\right)dV\left(-\frac{3}{2}\theta\right)\\ &\le \left[\max_{-1\le\theta\le 0}\left(e^{0.98\left(\frac{\theta+1}{2}\right)}\left(\frac{\theta+1}{2}\right)\right)\right]V\left(\frac{(\theta+1)^2}{4}\right)(-1,0) + \left[\max_{-1\le\theta\le 0}\left(e^{0.98\left(\frac{\theta}{4}\right)}\left(-\frac{\theta}{4}\right)\right)\right]V\left(-\frac{3}{2}\theta\right)(-1,0)\\ &= \frac{e^{\frac{0.98}{2}}}{2}\cdot\frac{1}{4} + \frac{e^{-\frac{0.98}{4}}}{4}\cdot\frac{3}{2} \cong 0.5 < 1.\end{aligned}$$

So, the zero solution of (1.1) is asymptotically stable.

In this example, stability analysis can be performed using Corollary 4.2 without using the characteristic equation (1.3). Indeed, if the conditions of (4.1), (4.2), and (4.3) are calculated, respectively, since $=\frac{1}{2}$, we get

$$\int_{-1}^{0} e^{(\theta+1)}d\left(\frac{(\theta+1)^2}{4}\right) + \int_{-1}^{0} e^{\frac{\theta}{2}}d\left(-\frac{3}{2}\theta\right) = \frac{1}{2}\int_{-1}^{0}\left[(\theta+1)e^{\theta+1} - 3e^{\frac{\theta}{2}}\right]d\theta \cong -0.68 > -2,$$

$$\int_{-1}^{0} e^{-(\theta+1)}d\left(\frac{(\theta+1)^2}{4}\right) + \int_{-1}^{0} e^{-\frac{\theta}{2}}d\left(-\frac{3}{2}\theta\right) = \frac{1}{2}\int_{-1}^{0}\left[(\theta+1)e^{-(\theta+1)} - 3e^{-\frac{\theta}{2}}\right]d\theta \cong -1.68 < 2,$$

and

$$\begin{aligned}&\int_{-1}^{0}\left(\frac{\theta+1}{2}\right)e^{(\theta+1)}dV\left(\frac{(\theta+1)^2}{4}\right) + \int_{-1}^{0}\left(-\frac{\theta}{4}\right)e^{-\frac{\theta}{2}}dV\left(-\frac{3}{2}\theta\right) \le \left[\max_{-1\le\theta\le 0}\left(\left(\frac{\theta+1}{2}\right)e^{(\theta+1)}\right)\right]V\left(\frac{(\theta+1)^2}{4}\right)(-1,0)\\ &+ \left[\max_{-1\le\theta\le 0}\left(\left(-\frac{\theta}{4}\right)e^{-\frac{\theta}{2}}\right)\right]V\left(-\frac{3}{2}\theta\right)(-1,0)\\ &= \frac{e}{2}\cdot\frac{1}{4} + \frac{\sqrt{e}}{4}\cdot\frac{3}{2} = \frac{1}{8}\left(e + 3\sqrt{e}\right) \cong 0.96 \le 1.\end{aligned}$$

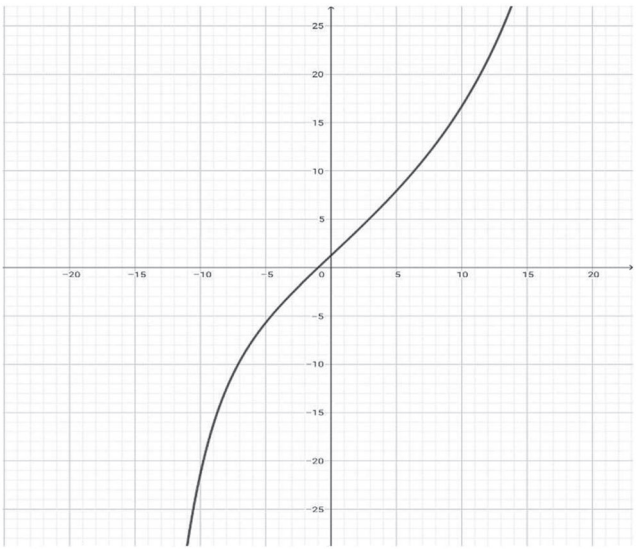


FIGURE 2 Location of the root of (5.2)

Thus, according to Lemma 4.1, it states that a real root must pass in the interval $(-2,2)$. Finally, from Corollary 4.2, we obtain

$$v(-1)+\eta(-1)=0+\frac{3}{2}>v(0)+\eta(0)=\frac{1}{4}+0,$$

and thus the zero solution is asymptotically stable.

Example 5.3. Consider Equation 1.5 for $a=b=\frac{1}{2}$ and $\mu(\theta)=-\frac{\theta}{2}$, namely, that

$$x'(t)=\int_{-\frac{1}{2}}^{\frac{1}{2}}x(t+\theta)d\left(-\frac{\theta}{2}\right).$$

Then, the above equation becomes

$$\begin{aligned}x'(t)&=\int_{-\frac{1}{2}}^{0}x(t+\theta)d\left(-\frac{\theta}{2}\right)+\int_{0}^{\frac{1}{2}}x(t+\theta)d\left(-\frac{\theta}{2}\right)=\int_{-\frac{1}{2}}^{0}x(t-(-\theta))d\left(-\frac{\theta}{2}\right)-\int_{-\frac{1}{2}}^{0}x(t+(-\theta))d\left(\frac{\theta}{2}\right)\\&=\int_{-1}^{0}x\left(t-\left(-\frac{\theta}{2}\right)\right)d\left(-\frac{\theta}{4}\right)+\int_{-1}^{0}x\left(t+\left(-\frac{\theta}{2}\right)\right)d\left(-\frac{\theta}{4}\right).\end{aligned}$$

The last equation here is a special case of Equation 1.1 for $r_1(\theta)=r_2(\theta)=-\frac{\theta}{2}$ and $v(\theta)=\eta(\theta)=-\frac{\theta}{4}$. In this case, the conditions of Lemma 4.3 is provided; that is, v and η are decreasing on $[-1,0]$, $r=\frac{1}{2}$, and from (4.1), we have

$$\int_{-1}^{0}e^{-\theta}d\left(-\frac{\theta}{4}\right)+\int_{-1}^{0}e^{\theta}d\left(-\frac{\theta}{4}\right)=-\frac{1}{4}\int_{-1}^{0}\left(e^{-\theta}+e^{\theta}\right)d\theta\cong-0.59>-2.$$

Therefore, one of the roots of the characteristic equation (1.3) is located in the interval $(-\infty,-2)$ and the other in the interval $(-2,0)$. Indeed, the characteristic equation (1.3) is

$$\begin{aligned}\lambda&=\int_{-1}^{0}e^{-\lambda\left(-\frac{\theta}{2}\right)}d\left(-\frac{\theta}{4}\right)+\int_{-1}^{0}e^{\lambda\left(-\frac{\theta}{2}\right)}d\left(-\frac{\theta}{4}\right)\\&=-\frac{1}{4}\int_{-1}^{0}\left(e^{\frac{\lambda\theta}{2}}+e^{-\frac{\lambda\theta}{2}}\right)d\theta=-\frac{1}{2\lambda}\left(e^{\frac{\lambda}{2}}-e^{-\frac{\lambda}{2}}\right).\end{aligned}\tag{5.3}$$

The characteristic function of (5.3) is

$$F_3(\lambda)=\lambda+\frac{1}{2\lambda}\left(e^{\frac{\lambda}{2}}-e^{-\frac{\lambda}{2}}\right),$$

and its graph is given in Figure 3.

In the graph of the F_3 function, it is seen that (5.3) has two roots. We see that $\lambda\cong-0.5$ and $\lambda\cong-11$ are roots of (5.3). Let $\lambda_0=-11$. From (2.2), we have

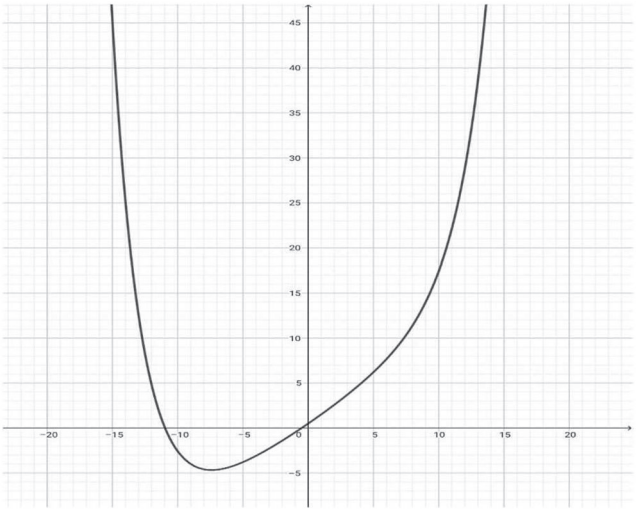


FIGURE 3 Locations of the roots of (5.3)

$$|\beta(-11)| = \left| \int_{-1}^{0} \left(-\frac{\theta}{2}\right) e^{-\frac{11\theta}{2}} d\left(-\frac{\theta}{4}\right) - \int_{-1}^{0} \left(-\frac{\theta}{2}\right) e^{\frac{11\theta}{2}} d\left(-\frac{\theta}{4}\right) \right| = \frac{1}{8} \left| \int_{-1}^{0} \frac{\theta}{2} \left(e^{-\frac{11\theta}{2}} - e^{\frac{11\theta}{2}}\right) d\theta \right| \cong 2.46 \leq \mu(-11).$$

So Theorem 3.1 for $\lambda_0 = -11$ cannot be applied. But, for $\lambda_0 = -0.5$, from (2.1) we get

$$\begin{aligned}
\mu(\lambda_0) = \mu\left(-\frac{1}{2}\right) &= \int_{-1}^{0} \left(-\frac{\theta}{2}\right) e^{-\frac{\theta}{4}} dV\left(-\frac{\theta}{4}\right) + \int_{-1}^{0} \left(-\frac{\theta}{2}\right) e^{\frac{\theta}{4}} dV\left(-\frac{\theta}{4}\right) \\
&\leq \left[\max_{-1 \leq \theta \leq 0} \left(-\frac{\theta}{2}\right) e^{-\frac{\theta}{4}}\right] V\left(-\frac{\theta}{4}\right)(-1,0) + \left[\max_{-1 \leq \theta \leq 0} \left(-\frac{\theta}{2}\right) e^{\frac{\theta}{4}}\right] V\left(-\frac{\theta}{4}\right)(-1,0) \\
&= \frac{e^{\frac{1}{4}}}{2} \cdot \frac{1}{4} + \frac{e^{-\frac{1}{4}}}{2} \cdot \frac{1}{4} \cong 0.26 < 1.
\end{aligned}$$

Then, for $\lambda_0 = -0.5$, the condition (2.1) of Theorem 3.1 is satisfied. So, the zero solution of (1.1) is asymptotically stable.

Example 5.4. Consider Equation 1.1 for $r_1(\theta) = r_2(\theta) = \frac{\theta+1}{2}$, $v(\theta) = \frac{3\theta}{4}$, and $\eta(\theta) = -\frac{3\theta^2}{4}$. In this case, the conditions of Lemma 4.4 are satisfied, that is, v and η are increasing on $[-1,0]$, $r = \frac{1}{2}$, and from (4.2), we have

$$\int_{-1}^{0} e^{-(\theta+1)} d\left(\frac{3\theta}{4}\right) + \int_{-1}^{0} e^{(\theta+1)} d\left(-\frac{3\theta^2}{4}\right) = \frac{3}{4} \int_{-1}^{0} \left(e^{-(\theta+1)} - 2\theta e^{(\theta+1)}\right) d\theta \cong 1.55 < 2.$$

Therefore, one of the roots of the characteristic equation (1.3) is located in the interval $(0,2)$ and the other in the interval $(2,\infty)$. Indeed, the characteristic equation (1.3) is

$$\lambda = \int_{-1}^{0} e^{-\lambda\left(\frac{\theta+1}{2}\right)} d\left(\frac{3\theta}{4}\right) + \int_{-1}^{0} e^{\lambda\left(\frac{\theta+1}{2}\right)} d\left(-\frac{3\theta^2}{4}\right) = \frac{3}{4} \int_{-1}^{0} \left(e^{-\lambda\left(\frac{\theta+1}{2}\right)} - 2\theta e^{\lambda\left(\frac{\theta+1}{2}\right)}\right) d\theta = \frac{3}{2\lambda} \left(\frac{4}{\lambda} e^{\frac{\lambda}{2}} - \frac{4}{\lambda} - 1 - e^{-\frac{\lambda}{2}}\right). \tag{5.4}$$

The characteristic function of (5.4) is

$$F_4(\lambda)=\lambda-\frac{3}{2\lambda}\left(\frac{4}{\lambda}e^{\frac{\lambda}{2}}-\frac{4}{\lambda}-1-e^{-\frac{\lambda}{2}}\right),$$

and its graph is given in Figure 4.

The graph of the function F_4 shows that two roots of (5.4) are passed and we see that $\lambda\cong 1.51$ and $\lambda\cong 10.64$ are roots of (5.4). Let's just take $\lambda_0=1.51$ from these two roots. We obtain

$$\begin{aligned}\mu(\lambda_0)&=\mu(1.51)\\&=\int_{-1}^{0}\left(\frac{\theta+1}{2}\right)e^{-1.51\left(\frac{\theta+1}{2}\right)}dV\left(\frac{3\theta}{4}\right)+\int_{-1}^{0}\left(\frac{\theta+1}{2}\right)e^{1.51\left(\frac{\theta+1}{2}\right)}dV\left(-\frac{3\theta^2}{4}\right)\le\left[\max_{-1\le\theta\le 0}\left(\frac{\theta+1}{2}\right)e^{-1.51\left(\frac{\theta+1}{2}\right)}\right]V\left(\frac{3\theta}{4}\right)(-1,0)\\&\quad+\left[\max_{-1\le\theta\le 0}\left(\frac{\theta+1}{2}\right)e^{1.51\left(\frac{\theta+1}{2}\right)}\right]V\left(-\frac{3\theta^2}{4}\right)(-1,0)\\&=\frac{3}{8}\left(e^{-1.51}+e^{1.51}\right)\cong 0.97<1.\end{aligned}$$

Then, for $\lambda_0=1.51$, the condition (2.1) of Theorem 3.1 is satisfied. So, the zero solution of (1.1) is unstable.

6 | A RESULT ON THE BEHAVIOR OF THE SOLUTIONS

Now, we will establish the following theorem.

Theorem 6.1. *Suppose that v is decreasing and η is increasing on $[-1,0]$. Let λ_0 be a real root of the characteristic equation (1.3) with the property 2.1 and let $\beta(\lambda_0)$ and $L(\lambda_0;\phi)$ be defined by (2.2) and (2.3), respectively. Let also λ_1 be a real root of the characteristic equation (1.3) with $\lambda_1<\lambda_0$. Then, for any $\phi\in C([-\|r_1\|,\|r_2\|],\mathbb{R})$, the solution x of (1.1)–(1.2) satisfies*

$$M_1(\lambda_0,\lambda_1;\phi)\le e^{-\lambda_1 t}\left(x(t)-\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right)\le M_2(\lambda_0,\lambda_1;\phi)\ \ for\ \ \forall t\ge 0, \tag{6.1}$$

where

$$M_1(\lambda_0,\lambda_1;\phi)=\min_{-\|r_1\|\le t\le\|r_2\|}\left\{e^{-\lambda_1 t}\left(\phi(t)-\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right)\right\}, \tag{6.2}$$

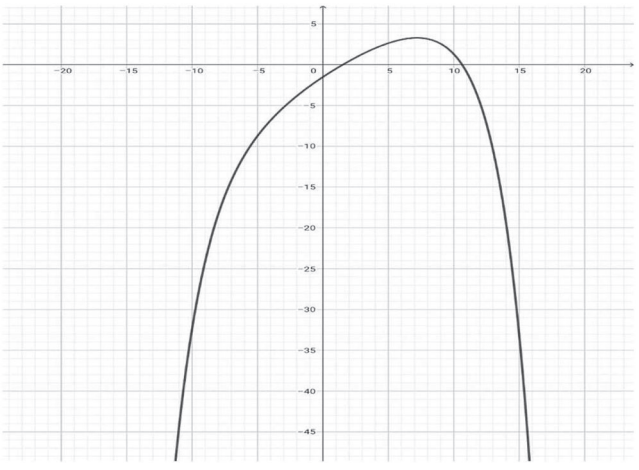


FIGURE 4 Locations of the roots of 5.4

and

$$M_2(\lambda_0,\lambda_1;\phi)=\max_{-\|r_1\|\le t\le\|r_2\|}\left\{e^{-\lambda_1 t}\left(\phi(t)-\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right)\right\}. \tag{6.3}$$

Proof. Let x be the solution of the initial problem (1.1)–(1.2) and consider the functions y and z defined as in the proof of Theorem 2.1, that is,

$$y(t)=e^{-\lambda_0 t}x(t) \text{ and } z(t)=y(t)-\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} \quad \text{for } t\in[-\|r_1\|,\infty).$$

Note that because of the fact that v is decreasing and η is increasing on $[-1,0]$, the property 2.1 can be written as $-1<\beta(\lambda_0)<0$, i.e. $0<1+\beta(\lambda_0)<1$. As it has been shown in the Theorem 2.1, the fact that x satisfies (1.1) for $t\ge 0$ is equivalent to the fact that z satisfies (2.9) for all $t\ge 0$.

Now we define

$$w(t)=e^{(\lambda_0-\lambda_1)t}z(t) \text{ for } t\ge -\|r_1\|.$$

Because of the definitions of y and z, we have the following expression for the function w:

$$w(t)=e^{-\lambda_1 t}\left\{x(t)-\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right\} \textit{ for } t\ge -\|r_1\|. \tag{6.4}$$

Furthermore, by the use of the function w, (2.9) can equivalently be written as

$$\begin{aligned} \mathrm{w(t)} = &-\int_{-1}^{0} \mathrm{e}^{-\lambda_0 \mathrm{r}_1(\theta)}\left(\int_{\mathrm{t}-\mathrm{r}_1(\theta)}^{\mathrm{t}} \mathrm{e}^{(\lambda_1-\lambda_0)(\mathrm{s}-\mathrm{t})}\mathrm{w(s)ds}\right)\mathrm{dv}(\theta) \\ &+\int_{-1}^{0} \mathrm{e}^{\lambda_0 \mathrm{r}_2(\theta)}\left(\int_{\mathrm{t}}^{\mathrm{t}+\mathrm{r}_2(\theta)} \mathrm{e}^{(\lambda_1-\lambda_0)(\mathrm{s}-\mathrm{t})}\mathrm{w(s)ds}\right)\mathrm{d}\eta(\theta), \end{aligned} \tag{6.5}$$

for $t\ge 0$ and (2.10) becomes

$$w(t)=e^{-\lambda_1 t}\left\{\phi(t)-\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right\} \textit{ for } t\in[-\|r_1\|,\|r_2\|]. \tag{6.6}$$

As solution x satisfies the initial condition (1.2), we can use (6.4) as well as the definition of $M_1(\lambda_0,\lambda_1;\phi)$ and $M_2(\lambda_0,\lambda_1;\phi)$ by (6.2) and (6.3), respectively, to see that

$$M_1(\lambda_0,\lambda_1;\phi)=\min_{-\|r_1\|\le t\le\|r_2\|} w(t) \text{ and } M_2(\lambda_0,\lambda_1;\phi)=\max_{-\|r_1\|\le t\le\|r_2\|} w(t). \tag{6.7}$$

In view of (6.4) and (6.7), the double inequality (6.1) can equivalently be written as follows:

$$\min_{-\|r_1\| \le s \le \|r_2\|} w(s) \le w(t) \le \max_{-\|r_1\| \le s \le \|r_2\|} w(s) \quad \text{for all } t \ge 0. \tag{6.8}$$

We have to prove that (6.8) holds. We will use the fact that w satisfies (6.5) for all $t \ge 0$ in order to show that (6.8) is valid. We restrict ourselves to proving that

$$w(t) \le \max_{-\|r_1\| \le s \le \|r_2\|} w(s) \quad \text{for every } t \ge 0. \tag{6.9}$$

The proof of the inequality

$$\min_{-\|r_1\| \le s \le \|r_2\|} w(s) \le w(t) \quad \text{for every } t \ge 0$$

can be obtained in a similar way, and so it is omitted. In the rest of the proof, we will establish (6.9). In order to so, we consider an arbitrary real number A with $w(t) \le \max_{-\|r_1\| \le s \le \|r_2\|} w(s) < A$, that is, with

$$w(t) < A \quad \text{for} \quad -\|r_1\| \le t \le 0. \tag{6.10}$$

We will show that

$$w(t) < A \ \text{ for } t \ge 0. \tag{6.11}$$

To this end, let us assume that (6.11) fails to hold. Then, because of (6.10), there exists a point $t_0 > 0$ so that

$$w(t) < A \,\text{for}\, t \in [-\|r_1\|, t_0) \cup (t_0, t_0 + \|r_2\|], \text{and } w(t_0) = A.$$

Thus, by using the assumption that v is decreasing and η is increasing on $[-1,0]$, from (6.5) we obtain

$$\begin{aligned}
A = w(t_0) &= -\int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} \left(\int_{t_0 - r_1(\theta)}^{t_0} e^{(\lambda_1 - \lambda_0)(s - t_0)} w(s) ds \right) dv(\theta) \\
&\quad + \int_{-1}^{0} e^{\lambda_0 r_2(\theta)} \left(\int_{t_0}^{t_0 + r_2(\theta)} e^{(\lambda_1 - \lambda_0)(s - t_0)} w(s) ds \right) d\eta(\theta) \\
&< A \left\{ -\int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} \left(\int_{t_0 - r_1(\theta)}^{t_0} e^{(\lambda_1 - \lambda_0)(s - t_0)} ds \right) dv(\theta) + \int_{-1}^{0} e^{\lambda_0 r_2(\theta)} \left(\int_{t_0}^{t_0 + r_2(\theta)} e^{(\lambda_1 - \lambda_0)(s - t_0)} ds \right) d\eta(\theta) \right\} \\
&= \frac{A}{\lambda_1 - \lambda_0} \left\{ -\int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} \left[1 - e^{-(\lambda_1 - \lambda_0) r_1(\theta)} \right] dv(\theta) + \int_{-1}^{0} e^{\lambda_0 r_2(\theta)} \left[e^{(\lambda_1 - \lambda_0) r_2(\theta)} - 1 \right] d\eta(\theta) \right\} \\
&= \frac{A}{\lambda_1 - \lambda_0} \left\{ -\int_{-1}^{0} \left(e^{-\lambda_0 r_1(\theta)} - e^{-\lambda_1 r_1(\theta)} \right) dv(\theta) + \int_{-1}^{0} \left(e^{\lambda_1 r_2(\theta)} - e^{\lambda_0 r_2(\theta)} \right) d\eta(\theta) \right\} \\
&= \frac{A}{\lambda_1 - \lambda_0} \left\{ -\int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} dv(\theta) - \int_{-1}^{0} e^{\lambda_0 r_2(\theta)} d\eta(\theta) + \int_{-1}^{0} e^{-\lambda_1 r_1(\theta)} dv(\theta) + \int_{-1}^{0} e^{\lambda_1 r_2(\theta)} d\eta(\theta) \right\} \\
&= \frac{A}{\lambda_1 - \lambda_0} (-\lambda_0 + \lambda_1) = A.
\end{aligned}$$

We have thus arrived at a contradiction and so (6.11) is true. Since (6.11) is satisfied for all real numbers A with $\max_{-\|r_1\| \le s \le \|r_2\|} w(s) < A$, it follows that (6.9) is always fulfilled. The proof of the Theorem 6.1 is complete. □

Corollary 6.2. *Assume that the conditions of Theorem 6.1 are satisfied. Then, for any $\phi \in C([-\|r_1\|, \|r_2\|], \mathbb{R})$, the solution x of (1.1)–(1.2) satisfies*

$$\lim_{t\to\infty}\left[e^{-\lambda_0 t}x(t)\right] = \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)},$$

where $\beta(\lambda_0)$ and $L(\lambda_0;\phi)$ are defined by (2.2) and (2.3), respectively.

Proof. We see that the double inequality in the conclusion of the Theorem 6.1 can equivalently be written in the following form:

$$M_1(\lambda_0,\lambda_1;\phi)e^{(\lambda_1-\lambda_0)t} \le e^{-\lambda_0 t}x(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} \le M_2(\lambda_0,\lambda_1;\phi)e^{(\lambda_1-\lambda_0)t}, \quad t \ge 0,$$

and hence, as $\lambda_1 < \lambda_0$, we get

$$\lim_{t\to\infty}\left[e^{-\lambda_0 t}x(t)\right] = \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)},$$

which establishes Corollary 6.2.

Moreover, this double inequality is also equivalent to the following one:

$$M_1(\lambda_0,\lambda_1;\phi)e^{\lambda_1 t} + \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t} \le x(t) \le M_2(\lambda_0,\lambda_1;\phi)e^{\lambda_1 t} + \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t},$$

for all $t \ge 0$ and we see

$$\lim_{t\to\infty} x(t) = 0,$$

provided if $\lambda_1 < \lambda_0 < 0$.

Below we will give a lemma about two different real roots of the characteristic equation (1.3).

Lemma 6.3. Let λ_0 be a real root of the characteristic equation (1.3) and let $\beta(\lambda_0)$ be defined by (2.2). Let also λ_1 be a real root of the characteristic equation (1.3) with $\lambda_1 < \lambda_0$. Assume that

$$\int_{-1}^{0} r_1^2(\theta)e^{-\lambda r_1(\theta)}dv(\theta) + \int_{-1}^{0} r_2^2(\theta)e^{\lambda r_2(\theta)}d\eta(\theta) < 0 \quad \text{for } \lambda \in (\lambda_1,\lambda_0). \tag{6.12}$$

Then $1+\beta(\lambda_0) > 0$.

Proof. Consider the real-valued function F defined by (4.5). The derivatives F' and F'' of F are given by (4.7) and (4.8), respectively. So, taking into account (6.12), we conclude that for $\lambda \in (\lambda_1,\lambda_0)$

$$F''(\lambda) > 0, \tag{6.13}$$

and consequently F is convex on (λ_1,λ_0).

On the other hand, since $F(\lambda_0)=F(\lambda_1)=0$, from Rolle's theorem it follows that $F'(c)=0$ for some point c such that $\lambda_1<c<\lambda_0$. But (6.13) implies that F' is positive on $(c,\lambda_0]$. Thus, we must have $F'(\lambda_0)>0$, that is,

$$\begin{aligned}F'(\lambda_0)&=1+\int_{-1}^{0} r_1(\theta)e^{-\lambda_0 r_1(\theta)}dv(\theta)-\int_{-1}^{0} r_2(\theta)e^{\lambda_0 r_2(\theta)}d\eta(\theta)\\&=1+\beta(\lambda_0)>0,\end{aligned}$$

and so the proof of the lemma is complete.

Example 6.4. Let's consider Example 5.1. In this example, the roots of the characteristic equation (5.1) are found as $\lambda=0$ and $\lambda\cong\mp 4.36$ and it is seen that condition (2.1) is fulfilled for $\lambda=0$. Also, v is decreasing and η is increasing on $[-1,0]$. Let's choose $\lambda_0=0$ and $\lambda_1=-4.36$. In this case, by applying Theorem 6.1, we obtain the following result:

For any continuous real-valued function ϕ on $[-1,1]$, the solution x of (1.1)–(1.2) satisfies

$$M_1(0,-4.36;\phi)\le e^{4.36t}\left(x(t)-\frac{L(0;\phi)}{1+\beta(0)}\right)\le M_2(0,-4.36;\phi)\quad\text{for all } t\ge 0.$$

Since

$$\begin{aligned}M_1(0,-4.36;\phi)&=\min_{-1\le t\le 1}\left\{e^{4.36t}\left(\phi(t)-\frac{L(0;\phi)}{1+\beta(0)}\right)\right\},\\M_2(0,-4.36;\phi)&=\max_{-1\le t\le 1}\left\{e^{4.36t}\left(\phi(t)-\frac{L(0;\phi)}{1+\beta(0)}\right)\right\},\end{aligned}$$

and

$$\beta(0)=\int_{-1}^{0}(-\theta)d\left(-\frac{\theta}{4}\right)-\int_{-1}^{0}(\theta+1)d\left(\frac{\theta}{4}+1\right)=-\frac{1}{4},$$

from the last double inequality, we have

$$\min_{-1\le t\le 1}\left\{e^{4.36t}\left(\phi(t)-\frac{L(0;\phi)}{3/4}\right)\right\}\le e^{4.36t}\left(x(t)-\frac{L(0;\phi)}{3/4}\right)\le\max_{-1\le t\le 1}\left\{e^{4.36t}\left(\phi(t)-\frac{L(0;\phi)}{3/4}\right)\right\},$$

for all $t\ge 0$, where

$$L(0;\phi)=\phi(0)-\frac{1}{4}\left[\int_{-1}^{0}\left(\int_{\theta}^{0}\phi(s)ds\right)d(\theta)+\int_{-1}^{0}\left(\int_{0}^{\theta+1}\phi(s)ds\right)d(\theta)\right].$$

Also, by applying Corollary 6.2, we get

$$\lim_{t\to\infty}x(t)=\frac{L(0;\phi)}{3/4}.$$

7 | THE SPECIAL CASE OF LINEAR AUTONOMOUS MIXED TYPE DIFFERENTIAL EQUATIONS

Our goal in this section is to give new results on asymptotic behavior and stability for a class of (1.4). Our results will be obtained via an appropriate solution of the corresponding characteristic equation.

Throughout this section, r and τ represent positive integers determined by

$$\mathrm{r}=\max\{r_i:\ i=1,...,p\}\ \text{and}\ \tau=\max\{\tau_j:\ j=1,...,q\}.$$

Along with the mixed type differential equation (1.4), we specify an *initial condition* of the form

$$x(t)=\theta(t),\quad -\varepsilon\le t\le\tau, \tag{7.1}$$

where the initial function θ is given a continuous real-valued function on the interval $[-\varepsilon,\tau]$ satisfying the *consistency condition*

$$\theta'(0)=\sum_{i=1}^{p}a_i\theta(-r_i)+\sum_{j=1}^{q}b_j\theta(\tau_j).$$

As is known, by a *solution* of (1.4), we mean a continuous function $x:[-\varepsilon,+\infty)\to\mathbb{R}$, which is differentiable on $[0,+\infty)$ and satisfies (1.4) for every $t\ge 0$.

If a solution of (1.4) is searched in the form $x(t)=e^{\lambda t}$ for $t\in\mathbb{R}$, the *characteristic equation* of (1.4) will be

$$\lambda=\sum_{i=1}^{p}a_ie^{-\lambda r_i}+\sum_{j=1}^{q}b_je^{\lambda\tau_j}. \tag{7.2}$$

The first result is the following theorem, which creates a basic asymptotic property for solutions of the mixed-type differential equation (1.4).

Theorem 7.1. *Let λ_0 be a real root of the characteristic Equation 7.2 with the property*

$$\widetilde{\mu}(\lambda_0)=\sum_{i=1}^{p}|\,a_i\,|\,r_ie^{-\lambda_0 r_i}+\sum_{j=1}^{q}|\,b_j\,|\,\tau_je^{\lambda_0\tau_j}<1, \tag{7.3}$$

and set

$$\widetilde{\beta}(\lambda_0)=\sum_{i=1}^{p}a_ir_ie^{-\lambda_0 r_i}-\sum_{j=1}^{q}b_j\tau_je^{\lambda_0\tau_j}. \tag{7.4}$$

Then, for every $\theta\in C([-\mathrm{r},\tau],\mathbb{R})$, the solution x of (1.4) and (7.1) satisfies

$$\lim_{t\to\infty}\left[e^{-\lambda_0 t}x(t)\right]=\frac{\widetilde{L}(\lambda_0;\theta)}{1+\widetilde{\beta}(\lambda_0)},$$

where

$$\widetilde{L}(\lambda_0;\theta)=\theta(0)+\sum_{i=1}^{p}a_i e^{-\lambda_0 r_i}\int_{-r_i}^{0}e^{-\lambda_0 s}\theta(s)ds-\sum_{j=1}^{q}b_j e^{\lambda_0\tau_j}\int_{0}^{\tau_j}e^{-\lambda_0 s}\theta(s)ds.$$

Note that property 7.3 guarantees that $1+\widetilde{\beta}(\lambda_0)>0$.

Note that we clearly see that $\lambda_0=0$ is a root of the (7.2) with the property 7.3 if and only if

$$\sum_{i=1}^{p}a_i+\sum_{j=1}^{q}b_j=0 \text{ and } \sum_{i=1}^{p}|a_i|r_i+\sum_{j=1}^{q}|b_j|\tau_j<1. \tag{7.5}$$

Thus, an application of Theorem 2.1 with $\lambda_0=0$ leads to the following corollary.

Corollary 7.2. *Let the conditions in (7.5) be satisfied. Then, for every $\theta\in C([-\varrho,\tau],\mathbb{R})$, the solution x of (1.4) and (7.1) satisfies*

$$\lim_{t\to\infty}x(t)=\frac{\theta(0)+\sum_{i=1}^{p}a_i\int_{-r_i}^{0}\theta(s)ds-\sum_{j=1}^{q}b_j\int_{0}^{\tau_j}\theta(s)ds}{1+\sum_{i=1}^{p}a_i r_i-\sum_{j=1}^{q}b_j\tau_j}.$$

Note that because of (7.5), the second condition ensures that $1+\sum_{i=1}^{p}a_i r_i-\sum_{j=1}^{q}b_j\tau_j>0$.

The second main result in this section is Theorem 7.3. This theorem makes an estimate for the solutions of the mixed-type differential equation (1.4), which leads in a stability result for the zero solution of Equation 1.4.

Theorem 7.3. *Let λ_0 be a real root of (7.2) with the property 7.3. Consider $\widetilde{\mu}(\lambda_0)$ and $\widetilde{\beta}(\lambda_0)$ be defined by (7.3) and (7.4), respectively. Then, for every $\theta\in C([-r,\tau],\mathbb{R})$, the solution x of (1.4) and (7.1) satisfies*

$$|x(t)|\le\left[\frac{(1+\widetilde{\mu}(\lambda_0))^2}{1+\widetilde{\beta}(\lambda_0)}+\widetilde{\mu}(\lambda_0)\right]\widetilde{N}(\lambda_0;\phi)e^{\lambda_0 t}\quad \text{for all } t\ge 0,$$

where

$$\widetilde{N}(\lambda_0;\theta)=\max_{-\varrho\le t\le\tau}\left|e^{-\lambda_0 t}\theta(t)\right|.$$

Moreover, the zero solution of (1.4) is stable if $\lambda_0=0$, asymptotically stable if $\lambda_0<0$, and unstable if $\lambda_0>0$.

In the following lemma, we give an important condition about the real roots of the characteristic equation (7.2).

Lemma 7.4. Assume that

$$\sum_{i=1}^{p}a_i e^{\frac{r_i}{\delta}}+\sum_{j=1}^{q}b_j e^{-\frac{\tau_j}{\delta}}>-\frac{1}{\delta},$$
$$\sum_{i=1}^{p}a_i e^{-\frac{r_i}{\delta}}+\sum_{j=1}^{q}b_j e^{\frac{\tau_j}{\delta}}<\frac{1}{\delta},$$

and

$$\sum_{i=1}^{p} |a_i| r_i e^{\frac{r_i}{\delta}} + \sum_{j=1}^{q} |b_j| \tau_j e^{\frac{\tau_j}{\delta}} \leq 1,$$

where $\delta = max\{r, \tau\}$. Then, in the interval $\left(-\frac{1}{\delta}, \frac{1}{\delta}\right)$, the characteristic equation (7.2) has a unique root λ_0, and this root satisfies the property 7.3.

In the following example, we will apply the stability criteria of Theorem 7.3. For simplicity of example, we consider the problem as follows:

Example 7.5. Consider

$$\begin{aligned} x'(t) &= ax(t-1) + bx(t+1), \quad t \geq 0, \\ x(t) &= \theta(t), \quad -1 \leq t \leq 1, \end{aligned} \tag{7.6}$$

where a and b are nonzero real numbers and $\theta \in C([-1,1],\mathbb{R})$.

The characteristic equation of (7.6) is

$$\lambda = ae^{-\lambda} + be^{\lambda}. \tag{7.7}$$

We will find the real roots of the characteristic equation (7.7) in all three cases below. Note that for each real root λ_0 of the characteristic equation (7.7), from (7.3), it becomes

$$\widetilde{\mu}(\lambda_0) = |a| e^{-\lambda} + |b| e^{\lambda} < 1. \tag{7.8}$$

Case 1. Choose $a = -\frac{1}{3}$ and $b = \frac{1}{3}$. Then, we have $\lambda_1 = 0$ and $\lambda_2 \cong -1.62$, which are real roots of (7.7). Let's just take $\lambda_0 = 0$ from these two roots. From (7.8), we obtained $\widetilde{\mu}(0) = \left|-\frac{1}{3}\right| + \left|\frac{1}{3}\right| = \frac{2}{3} < 1$. Thus, the condition of Theorem 7.3 is satisfied and the zero solution of (7.6) is stable.

Case 2. Choose $a = \frac{1}{4e}$ and $b = -\frac{(2\sqrt{e}+1)}{4}$. We see that $\lambda = -\frac{1}{2}$ is unique real root of (7.7). Then for $\lambda_0 = -\frac{1}{2}$, we get

$$\widetilde{\mu}\left(-\frac{1}{2}\right) = \left|\frac{1}{4e}\right|\sqrt{e} + \left|-\frac{(2\sqrt{e}+1)}{4}\right|\frac{1}{\sqrt{e}} = \frac{1+\sqrt{e}}{2\sqrt{e}} \cong 0.803 < 1.$$

Thus, the condition of Theorem 7.3 is satisfied and the zero solution of (7.6) is asymptotically stable.

Case 3. Choose $a = \frac{\sqrt{e}-1}{2}$ and $b = \frac{1}{2e}$. We see that $\lambda_1 = \frac{1}{2}$ and $\lambda_2 \cong 2.67$ are real roots of (7.7). Let's just take $\lambda_0 = \frac{1}{2}$ from these two roots. Then we have

$$\widetilde{\mu}\left(\frac{1}{2}\right) = \left|\frac{\sqrt{e}-1}{2}\right|\frac{1}{\sqrt{e}} + \left|\frac{1}{2e}\right|\sqrt{e} = \frac{1}{2} < 1.$$

Thus, the condition of Theorem 7.3 is satisfied and the zero solution of (7.6) is unstable.

8 | CONCLUSIONS

In this paper, first, we prove a basic asymptotic criterion for the solutions of the mixed type differential equation 1.1. Second, a useful exponential estimate for the solutions is established and a stability criterion is given. These results are obtained by the use of a real root (with an appropriate property) of the corresponding characteristic equation 1.3. Third,

an important result is obtained for the behavior of the solutions by making use of two appropriate real roots of the characteristic equation. Five examples of these results are given. Also useful lemmas about the roots of 1.3 are obtained. Finally, we also presented the application of the obtained results in the special case of constant coefficients and gave three different cases in one example. The real roots used in this paper play an important role in determining the results.

We will be particularly concerned with the possibility of generalizing our results in the case of first-order linear mixed (advanced - delay) differential equations with variable delays and coefficients. This seems easy to achieve if the delays are variable and bounded. However, the general situation with variable delays seems somewhat more difficult. It will also be interesting to generalize our results for first-order linear inhomogeneous mixed (advanced - delay) differential equations with periodic coefficients.

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AUTHOR CONTRIBUTIONS

Ali Fuat Yeniçerioğlu: Investigation. **Cüneyt Yazıcı:** Investigation. **Sandra Pinelas:** Investigation.

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