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A new characterization of (P, Q)-Lucas polynomial coefficients of the bi-univalent function class associated with q-analogue of Noor integral operator

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Abstract
In this study, by using Lucas polynomials, subordination and q-analogue of Noor integral operator, we will introduce an interesting new class $Q^{q,\mu}(\tau, \alpha; x)$ of bi-univalent functions. Also we will obtain (P, Q)-Lucas polynomial coefficient estimates and Fekete–Szegö inequalities for this new class.

Keywords (P, Q)-Lucas polynomials · q-analogue of Noor integral operator · Coefficient bounds · Bi-univalent functions

Mathematics Subject Classification 30C45 · 30C50

1 Introduction

Some special polynomials, (such as Chebyshev polynomials, Fibonacci polynomials, Lucas polynomials, Jacobsthal–Lucas polynomials, Lucas–Lehmer polynomials, Pell polynomials, Fermat–Lucas polynomials, orthogonal polynomials) and their generalizations are gained importance in very different branches such as physics, engineering, architecture, nature, art, number theory, combinatorics and numerical analysis. In very recent years, these polynomials have been studied from a theoretical point (see [19, 20, 22, 25, 26, 28, 30, 34, 41, 42])

Definition 1 [26] Let $P(x)$ and $Q(x)$ are polynomials with real coefficients. The (P, Q) Lucas polynomials $\mathcal{L}_{P,Q,n}(x)$ are defined by recurrence relation

$$\mathfrak{L}_{P,Q,n}(x) = P(x)\mathfrak{L}_{P,Q,n-1}(x) + Q(x)\mathfrak{L}_{P,Q,n-2}(x) \quad (n \geq 2), \tag{1}$$

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from which the first few Lucas polynomials can be found as

$$\begin{aligned}\mathfrak{L}_{P,Q,0}(x) &= 2,\\ \mathfrak{L}_{P,Q,1}(x) &= P(x),\\ \mathfrak{L}_{P,Q,2}(x) &= P^2(x) + 2Q(x),\\ \mathfrak{L}_{P,Q,3}(x) &= P^3(x) + 3\mathcal{P}(x)Q(x).\end{aligned} \tag{2}$$

Some Special cases of the $\mathfrak{L}_{P,Q,n}(x)$ with the given initial conditions are given by following:

1 For $P(x) = x$ and $Q(x) = 1$, $\mathfrak{L}_{\mathcal{P},Q,n}(x)$ are the Lucas polynomials $\mathfrak{L}_n(x)$;
2 For $P(x) = 2x$ and $Q(x) = 1$, $\mathfrak{L}_{P,Q,n}(x)$ are the Pell–Lucas polynomials $\mathfrak{D}_n(x)$;
3 For $P(x) = 1$ and $Q(x) = 2x$, $\mathfrak{L}_{P,Q,n}(x)$ are the Jacobsthal–Lucas polynomials $\mathfrak{j}_n(x)$;
4 For $P(x) = 3x$ and $Q(x) = -2$, $\mathfrak{L}_{P,Q,n}(x)$ are the Fermat–Lucas polynomials $\mathfrak{f}_n(x)$;
5 For $P(x) = 2x$ and $Q(x) = -1$, $\mathfrak{L}_{P,Q,n}(x)$ are the Chebyshev polynomials first kind $T_n(x)$.

Definition 2 [26] Let $\mathcal{G}_{\{\mathfrak{L}_n(x)\}}(z)$ be the generating function of the (P, Q) Lucas polynomial sequence $\mathfrak{L}_{P,Q,n}(x)$. Then

$$\mathcal{G}_{\{\mathfrak{L}_n(x)\}}(z) = \sum_{n=0}^{\infty} \mathfrak{L}_{P,Q,n}(x) z^n = \frac{2 - P(x)z}{1 - P(x)z - Q(x)z^2}. \tag{3}$$

We know that the generating function $\mathcal{G}_{\{\mathfrak{L}_n(x)\}}(z)$ is a convergence formal series.

Let $\mathcal{A}$ be the family of functions f of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{4}$$

which are analytic in the open unit disc $\mathfrak{U} = \{z : |z| < 1\}$ and normalized under the conditions given by $f(0) = 0 = f'(0) - 1$. Let $S = \{f \in \mathcal{A} : f \text{ is univalent in } \mathfrak{U}\}$.

According to the Koebe One-Quarter Theorem [17] every function $f \in S$ has an inverse f^{-1} which satisfies the following conditions:

$$f^{-1}(f(z)) = z \quad (z \in \mathfrak{U})$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right), \quad (w \in \mathfrak{U})$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \tag{5}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathfrak{U}$ if both f and f^{-1} are univalent in $\mathfrak{U}$. Let Σ denote the class of bi-univalent functions in $\mathfrak{U}$ given by (4). For several interesting examples in the class, see [38]. See also [2–13, 21, 23, 24, 27, 35–37].

For analytic functions f and g in $\mathfrak{U}$, f is said to be subordinate to g if there exists an analytic function w such that

$$w(0) = 0,\ |w(z)| < 1 \text{ and } f(z) = g(w(z)).$$

This subordination is denoted by

$$f(z) \prec g(z). \tag{6}$$

In particular, when g is univalent in $\mathfrak{U}$,

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad and \quad f(\mathfrak{U}) \subset g(\mathfrak{U}) \quad (z \in \mathfrak{U}).$$

After Srivastava used the basic (or q-)hypergeometric functions in a book [39], the theory of univalent functions was characterized by the concept of q-calculus. For simplicity, we provide some essential definitions of q-calculus that are used in this paper. Next, we recall some identities of fractional q-calculus operators of complex valued function f:

Definition 3 [39] For $q \in (0, 1)$, the q-derivative of function $f \in \mathcal{A}$ is defined by

$$\partial_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, z \neq 0 \tag{7}$$

and

$$\partial_q f(0) = f'(0).$$

Thus we have

$$\partial_q f(z) = 1 + \sum_{k=2}^{\infty} [k, q]\, a_k z^{k-1} \tag{8}$$

where $[k, q]$ is given by

$$[k, q] = \frac{1 - q^k}{1 - q}, \quad [0, q] = 0 \tag{9}$$

and the q-fractional is defined by

$$[k, q]! = \begin{cases} \prod_{n=1}^{k} [n, q]\,, & k \in \mathbb{N} \\ 1, & k = 0 \end{cases}. \tag{10}$$

Also, the q-generalized Pochhammer symbol for $\mathfrak{p} \geq 0$ is given by

$$[\mathfrak{p}, q]_k = \begin{cases} \prod_{n=1}^{k} [\mathfrak{p} + n - 1, q]\,, & k \in \mathbb{N} \\ 1, & k = 0 \end{cases}. \tag{11}$$

As $q \to 1^-$, then we get $[k, q] \to k$. Thus, if we choose the function $g(z) = z^k$, while $q \to 1^-$, then we have

$$\partial_q g(z) = \partial_q z^k = [k, q]\, z^{k-1} = g'(z),$$

where g' is the ordinary derivative.

Newly, Arif et al. [14] defined the function $F^{-1}_{q,\mu+1}(z)$ given by the following relation

$$F^{-1}_{q,\mu+1}(z) * F_{q,\mu+1}(z) = z\partial_q f(z), \quad (\mu > -1) \tag{12}$$

where

$$F_{q,\mu+1}(z) = z + \sum_{k=2}^{\infty} \frac{[\mu + 1, q]_{k-1}}{[k-1, q]!} z^k, \quad z \in \mathfrak{U}. \tag{13}$$

Since the series defined in (13) is convergent absolutely in $z \in \mathfrak{U}$, by using the definition of q-derivative through convolution, we now define the integral operator $\zeta^{\mu}_q : \mathfrak{U} \to \mathfrak{U}$ by

$$\zeta^{\mu}_q f(z) = F^{-1}_{q,\mu+1}(z) * f(z) = z + \sum_{k=2}^{\infty} \phi_{k-1} a_k z^k, \quad (z \in \mathfrak{U}) \tag{14}$$

where

$$\phi_{k-1} = \frac{[k,q]!}{[\mu+1,q]_{k-1}}. \tag{15}$$

We note that

$$\zeta_q^0 f(z) = z\partial_q f(z),\ \zeta_q^1 f(z) = f(z) \tag{16}$$

also

$$\lim_{q\to 1^-} \zeta_q^\mu f(z) = z + \sum_{k=2}^{\infty} \frac{k!}{(\mu+1)_{k-1}} a_k z^k. \tag{17}$$

This shows that, by taking $q \to 1^-$, the operator defined in (14) reduces to the well known Noor integral operator studied in [31, 32] . Also for more details on the q-analogue of differential and integral operators, see for example, [1, 7, 24].

In 1933, Fekete and Szegö [18] obtained a sharp bound of the functional $\left|a_3 - \delta a_2^2\right|$, with real $\delta(0 \le \delta \le 1)$ for a univalent function f. Since then, the problem of finding the sharp bounds for this functional of any compact family of functions or $f \in \mathcal{A}$ with any complex δ is known as the classical Fekete–Szegö problem or inequality. The equality is valid for the Koebe function. The sharp upper bound for Fekete–Szegö functional was found by Keogh and Merkes [23] for some subclasses of univalent function classes. Recently, coefficient estimates and famous celebrated Fekete–Szegö inequality were studied for the class of univalent functional defined by some special functions [15, 16, 29, 33].

The main goal of this study is to define a new class $Q^{q,\mu}(\tau,\alpha;x)$ including q-analogue of Noor integral operator. Then we derived $|a_2|$ and $|a_3|$ coefficients by using (P, Q) Lucas polynomials for this class. Furthermore, we also solved Fekete–Szegö problem for the class $Q^{q,\mu}(\tau,\alpha;x)$.

For these purpose, we initially presented some general explanations about bi-univalent functions, q-analogue of Noor integral operator and (P, Q) Lucas polynomials.

Finally for some different values of parameters, we obtain remarks for different classes and corollaries for initial coefficients and Fekete–Szeg ö inequality.

2 The class $Q^{q,\mu}(\tau, \alpha; x)$ and the Fekete–Szegö inequality

In this part, we will introduce the bi-univalent function class $Q^{q,\mu}(\tau,\alpha;x)$ involving q-analogue of Noor integral operator. Then, we will obtain the (P, Q) Lucas polynomial coefficients for $|a_2|$ and $|a_3|$. We also determine Fekete–Szegö problem for this new class.

Definition 4 The function f is said to be in the class $Q^{q,\mu}(\tau,\alpha;x)$ if the following conditions are satisfied:

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^\mu f(z)}{z} + \alpha\partial_q(\zeta_q^\mu f(z)) - 1\right] \prec \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(z) - 1 \tag{18}$$

and

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^\mu g(w)}{w} + \alpha\partial_q(\zeta_q^\mu g(w)) - 1\right] \prec \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(w) - 1 \tag{19}$$

where $(z, w \in \mathfrak{U}, \mu > -1,\ 0 < q < 1,\ \tau > 0, \alpha \ge 0)$ and $g = f^{-1}$ is given by (5) .

Remark 1 By taking $\mu = 1$ in Definition 4, one can easily see that a function $f \in \Sigma$ be in $Q^{q,1}(\tau, \alpha; x)$ if the following conditions hold true:

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{f(z)}{z} + \alpha\partial_q f(z) - 1\right] \prec \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(z) - 1$$

and

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{g(w)}{w} + \alpha\partial_q g(w) - 1\right] \prec \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(w) - 1$$

where $g = f^{-1}$ is given by (5).

Remark 2 By taking $\mu = 0$ in Definition 4 one can easily see that a function $f \in \Sigma$ be in $Q^{q,0}(\tau, \alpha; x)$ if the following conditions hold true:

$$1 + \frac{1}{\tau}\left[(1-\alpha)\partial_q f(z) + \alpha z\partial_q^2 f(z) - 1\right] \prec \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(z) - 1$$

and

$$1 + \frac{1}{\tau}\left[(1-\alpha)\partial_q g(w) + \alpha z\partial_q^2 g(w) - 1\right] \prec \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(w) - 1$$

where $g = f^{-1}$ is given by (5)

Also choosing different values for q, μ, τ and α in Definition 4, we have following new classes:

Remark 3 By taking $q \to 1^-$, one can easily see that a function $f \in \Sigma$ be in $Q^{\mu}(\tau, \alpha; x)$ if the following conditions hold true:

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{\zeta^{\mu} f(z)}{z} + \alpha(\zeta^{\mu} f(z))' - 1\right] \prec \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(z) - 1$$

and

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{\zeta^{\mu} g(w)}{w} + \alpha(\zeta^{\mu} g(w))' - 1\right] \prec \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(w) - 1,$$

where $g = f^{-1}$ is given by (5) .

Remark 4 Upon setting $q \to 1^-$ and for $\tau = 1$, one can easily see that a function $f \in \Sigma$ is in

$$Q^{\mu}(1, \alpha; x) = Q^{\mu}(\alpha; x)$$

if the following conditions hold true:

$$(1-\alpha)\frac{\zeta^{\mu} f(z)}{z} + \alpha(\zeta^{\mu} f(z))' \prec \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(z) - 1$$

and

$$(1-\alpha)\frac{\zeta^{\mu} g(w)}{w} + \alpha(\zeta^{\mu} g(w))' \prec \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(w) - 1,$$

where $g = f^{-1}$ is given by (5) .

Remark 5 Upon setting $q \to 1^-$, for $\tau = 1$ and $\alpha = 1$, one can easily see that a function $f \in \Sigma$ is in

$$Q^{\mu}(1, 1; x) = Q^{1}(x)$$

if the following conditions hold true:

$$(\zeta^{\mu} f(z))' \prec \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(z) - 1$$

and

$$(\zeta^{\mu} g(w))' \prec \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(w) - 1,$$

where $g = f^{-1}$ is given by (5) .

Remark 6 Upon setting $q \to 1^-$, for $\tau = 1$, $\alpha = 1$ and $\mu = 1$, one can easily see that a function $f \in \Sigma$ is in

$$Q(1, 1; x) = Q^1(x)$$

if the following conditions hold true:

$$f'(z) \prec \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(z) - 1$$

and

$$g'(w) \prec \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(w) - 1,$$

where $g = f^{-1}$ is given by (5) . The class $\Sigma'(x)$ was investigated and studied by Alamoush [6].

Remark 7 Upon setting $q \to 1^-$, for $\tau = 1$, $\alpha = 1$ and $\mu = 0$, one can easily see that a function $f \in \Sigma$ is in

$$Q^0(1, 1; x) = Q(x)$$

if the following conditions hold true:

$$\left(zf'(z)\right)' \prec \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(z) - 1$$

and

$$\left(wg'(w)\right)' \prec \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(w) - 1,$$

where $g = f^{-1}$ is given by (2) .

Theorem 1 *Let f given by (4) be in the class $Q^{q,\mu}(\tau, \alpha; x)$. Then,*

$$|a_2| \leq \frac{\tau\,|P(x)|\,\sqrt{|P(x)|}}{\sqrt{\left|\left\{\tau(1+\alpha q+\alpha q^2)\phi_2 - (1+\alpha q)^2\,\phi_1^2\right\} P^2(x) - 2\,(1+\alpha q)^2\,\phi_1^2 Q(x)\right|}} \tag{20}$$

and

$$|a_3| \leq \frac{\tau^2 P^2(x)}{(1+\alpha q)^2\phi_1^2} + \frac{|\tau|\,|P(x)|}{(1+\alpha q+\alpha q^2)\phi_2}. \tag{21}$$

Proof Let $f \in Q^{q,\mu}(\tau, \alpha; x)$. Then from Definition 4, for some analytic functions Ω, Λ such that $\Omega(0) = \Lambda(0) = 0$ and $|\Omega(z)| < 1$, $|\Lambda(w)| < 1$ for all $z, w \in f$, we can write

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu} f(z)}{z} + \alpha\,\partial_q(\zeta_q^{\mu} f(z)) - 1\right] = \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(\Omega(z)) - 1 \tag{22}$$

and

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu} g(w)}{w} + \alpha\,\partial_q(\zeta_q^{\mu} g(w)) - 1\right] = \mathcal{G}_{\{\mathfrak{L}_{P,Q,n}(x)\}}(\Lambda(w)) - 1 \tag{23}$$

or equivalently

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu}f(z)}{z}+\alpha\partial_q(\zeta_q^{\mu}f(z))-1\right]$$
$$=-1+\mathfrak{L}_{P,Q,0}(x)+\mathfrak{L}_{P,Q,1}(x)\Omega(z)+\mathfrak{L}_{P,Q,2}(x)\Omega^2(z)+\cdots \tag{24}$$

and

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu}g(w)}{w}+\alpha\partial_q(\zeta_q^{\mu}g(w))-1\right]$$
$$=-1+\mathfrak{L}_{P,Q,0}(x)+\mathfrak{L}_{P,Q,1}(x)\Lambda(w)+\mathfrak{L}_{P,Q,2}(x)\Lambda^2(w)+\cdots. \tag{25}$$

From the equalities (24)and (25), we obtain that

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu}f(z)}{z}+\alpha\partial_q(\zeta_q^{\mu}f(z))-1\right]$$
$$=1+\mathfrak{L}_{P,Q,1}(x)s_1z+\left[\mathfrak{L}_{P,Q,1}(x)s_2+\mathfrak{L}_{P,Q,2}(x)s_1^2\right]z^2+\cdots \tag{26}$$

and

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu}g(w)}{w}+\alpha\partial_q(\zeta_q^{\mu}g(w))-1\right]=1+\mathfrak{L}_{P,Q,1}(x)r_1w$$
$$+\left[\mathfrak{L}_{P,Q,1}(x)r_2+\mathfrak{L}_{P,Q,2}(x)r_1^2\right]w^2+\cdots \tag{27}$$

It is known before that if for $z, w \in f$,

$$\Omega(z)=\left|\sum_{i=1}^{n}s_iz^i\right|<1$$

and

$$\Lambda(w)=\left|\sum_{i=1}^{n}r_iw^i\right|<1$$

than

$$|s_i|<1 \tag{28}$$

and

$$|r_i|<1 \tag{29}$$

where $i \in \mathbb{N}$. So, comparing the corresponding coefficients in (26) and eqrefeq27, we get

$$\frac{1}{\tau}(1+\alpha q)\,\phi_1a_2=\mathfrak{L}_{P,Q,1}(x)s_1, \tag{30}$$

$$\frac{1}{\tau}\left[(1+\alpha q+\alpha q^2)\phi_2a_3\right]=\mathfrak{L}_{P,Q,1}(x)s_2+\mathfrak{L}_{P,Q,2}(x)s_1^2, \tag{31}$$

$$-\frac{1}{\tau}(1+\alpha q)\,\phi_1a_2=\mathfrak{L}_{P,Q,1}(x)r_1, \tag{32}$$

$$\frac{1}{\tau}\left[(1+\alpha q+\alpha q^2)\phi_2\left(2a_2^2-a_3\right)\right]=\mathfrak{L}_{P,Q,1}(x)r_2+\mathfrak{L}_{P,Q,2}(x)r_1^2. \tag{33}$$

From (30) and (32)

$$s_1=-r_1, \tag{34}$$

$$2\left[\frac{1}{\tau}\left(1+\alpha q\right)\phi_1\right]^2 a_2^2 = \mathfrak{L}_{P,Q,1}^2(x)\left(s_1^2+r_1^2\right). \tag{35}$$

Adding (31) and (33) we get

$$\frac{2}{\tau}(1+\alpha q+\alpha q^2)\phi_2 a_2^2 = \mathfrak{L}_{P,Q,1}(x)\left(s_2+r_2\right)+\mathfrak{L}_{P,Q,2}(x)\left(s_1^2+r_1^2\right) \tag{36}$$

By using (35) in (36) we have

$$2\left[\tau\mathfrak{L}_{P,Q,1}^2(x)(1+\alpha q+\alpha q^2)\phi_2-\mathfrak{L}_{P,Q,2}(x)\left(1+\alpha q\right)^2\phi_1^2\right]a_2^2=\tau^2\mathfrak{L}_{P,Q,1}^3(x)\left(s_2+r_2\right) \tag{37}$$

which gives

$$|a_2|\leq\frac{\tau\left|P(x)\right|\sqrt{\left|P(x)\right|}}{\sqrt{\left|\left\{\tau(1+\alpha q+\alpha q^2)\phi_2-\left(1+\alpha q\right)^2\phi_1^2\right\}P^2(x)-2\left(1+\alpha q\right)^2\phi_1^2Q(x)\right|}}. \tag{38}$$

Also, by subtracting (31) from (33) , we get

$$2(1+\alpha q+\alpha q^2)\phi_2\left(a_3-a_2^2\right)=\tau\left[\mathfrak{L}_{P,Q,1}(x)\left(s_2-r_2\right)+\mathfrak{L}_{P,Q,2}(x)\left(s_1^2-r_1^2\right)\right]. \tag{39}$$

Then, by using (34) and (35) in (39), we have

$$a_3=\frac{\tau^2\mathfrak{L}_{P,Q,1}^2(x)\left(s_1^2+r_1^2\right)}{2(1+\alpha q)^2\phi_1^2}+\frac{\tau\mathfrak{L}_{P,Q,1}(x)\left(s_2-r_2\right)}{2(1+\alpha q+\alpha q^2)\phi_2}$$

and by the help of (28) and (29) and taking modules, we conclude that

$$|a_3|\leq\frac{\tau^2P^2(x)}{(1+\alpha q)^2\phi_1^2}+\frac{|\tau|\left|P(x)\right|}{(1+\alpha q+\alpha q^2)\phi_2}.$$

□

The next theorem gives us the Fekete–Szegö inequality :

Theorem 2 *For $\delta\in\mathbb{R}$, let f given by (1) be in the class $Q^{q,\mu}(\tau,\alpha;x)$. Then,*

$$\left|a_3-\delta a_2^2\right|\leq\begin{cases}\frac{|\tau||P(x)|}{(1+\alpha q+\alpha q^2)\phi_2} & ,\ 0\leq|t(\delta;x)|\leq\frac{|\tau|}{2(1+\alpha q+\alpha q^2)\phi_2}\\ 2\left|P(x)\right|\left|t(\delta;x)\right| & ,\quad |t(\delta;x)|\geq\frac{|\tau|}{2(1+\alpha q+\alpha q^2)\phi_2}\end{cases} \tag{40}$$

where

$$t(\delta;x)=\frac{(1-\delta)\,\mathfrak{L}_{P,Q,1}^2(x)}{2\left[\mathfrak{L}_{P,Q,1}^2(x)(1+2\zeta)(1+2\delta)^n-\mathfrak{L}_{P,Q,2}(x)(1+\zeta)^2(1+\delta)^{2n}\right]}. \tag{41}$$

Proof From the equations (37) and (39) , we get

$$\begin{aligned} a_3 - \delta a_2^2 &= (1-\delta)\left[\frac{\tau^2 \mathfrak{L}^3_{P,Q,1}(x)\,(s_2+r_2)}{2\tau(1+\alpha q+\alpha q^2)\phi_2 \mathfrak{L}^2_{P,Q,1}(x) - 2(1+\alpha q)^2\phi_1^2 \mathfrak{L}_{P,Q,2}(x)}\right] \\ &\quad + \frac{\tau \mathfrak{L}_{P,Q,1}(x)\,(s_2-r_2)}{2(1+\alpha q+\alpha q^2)\phi_2} \\ &= \mathfrak{L}_{P,Q,1}(x)\left[\left(t(\delta;x) + \frac{\tau}{2(1+\alpha q+\alpha q^2)\phi_2}\right)s_2 + \left(t(\delta;x) - \frac{\tau}{2(1+\alpha q+\alpha q^2)\phi_2}\right)s_2\right], \end{aligned}$$

where

$$t(\delta;x) = \frac{\tau^2\,(1-\delta)\,\mathfrak{L}^2_{P,Q,1}(x)}{2\left[\tau \mathfrak{L}^2_{P,Q,1}(x)(1+\alpha q+\alpha q^2)\phi_2 - \mathfrak{L}_{P,Q,2}(x)(1+\alpha q)^2\phi_1^2\right]}.$$

By using equalities in Eq. (2), we complete the proof. □

3 Corollaries and consequences

An urgent and important corollary of the Theorems 1 and 2 for $q \to 1^-$ is asserted by Corollary 1 below.

Corollary 1 *Let the function $f \in Q^{\mu}(\tau,\alpha;x)$ be of the form (1). Also assume that $\delta \in \mathbb{R}$. Then,*

$$|a_2| \le \frac{|\tau|\,|P(x)|\sqrt{|P(x)|}}{\sqrt{\left|\left\{\tau(1+2\alpha)\phi_2 - (1+\alpha)^2\phi_1^2\right\}P^2(x) - 2(1+\alpha)^2\phi_1^2 Q(x)\right|}},$$

$$|a_3| \le \frac{\tau^2 P^2(x)}{(1+\alpha)^2\phi_1^2} + \frac{|\tau|\,|P(x)|}{(1+2\alpha)\phi_2}$$

and

$$\left|a_3 - \delta a_2^2\right| \le \begin{cases} |\tau|\,\frac{|P(x)|}{(1+2\alpha)\phi_2} & , \ 0 \le |t(\delta;x)| \le \frac{|\tau|}{2(1+2\alpha)\phi_2} \\ 2\,|P(x)|\,|t(\delta;x)| & , \ |t(\delta;x)| \ge \frac{|\tau|}{2(1+2\alpha)\phi_2} \end{cases}.$$

Choosing $\tau = 1$ and $q \to 1^-$ in Theorems 1 and 2, we get Corollary 2.

Corollary 2 *Let the function $f \in Q^{\mu,1}(\alpha,1;x)$ be of the form (1). Also assume that $\delta \in \mathbb{R}$. Then,*

$$|a_2| \le \frac{|P(x)|\sqrt{|P(x)|}}{\sqrt{\left|\left\{(1+2\alpha)\phi_2 - (1+\alpha)^2\phi_1^2\right\}P^2(x) - 2(1+\alpha)^2\phi_1^2 Q(x)\right|}},$$

$$|a_3| \le \frac{P^2(x)}{(1+\alpha)^2\phi_1^2} + \frac{|P(x)|}{(1+2\alpha)\phi_2}$$

and

$$\left|a_3 - \delta a_2^2\right| \le \begin{cases} \frac{|P(x)|}{(1+2\alpha)\phi_2} & , \ 0 \le |t(\delta;x)| \le \frac{1}{2(1+2\alpha)\phi_2} \\ 2\,|P(x)|\,|t(\delta;x)| & , \ |t(\delta;x)| \ge \frac{1}{2(1+2\alpha)\phi_2} \end{cases}.$$

Choosing $\alpha = 1$, $\tau = 1$ and $q \to 1^-$ in Theorems 1 and 2, we get Corollary 3.

Corollary 3 *Let the function* $f \in Q^{\mu}(1,1;x) = Q^{\mu}(x)$ *be of the form* (4). *Also assume that* $\delta \in \mathbb{R}$. *Then,*

$$|a_2| \leq \frac{|P(x)|\sqrt{|P(x)|}}{\sqrt{\left|\left\{3\phi_2 - 4\phi_1^2\right\}P^2(x) - 8\phi_1^2 Q(x)\right|}},$$

$$|a_3| \leq \frac{P^2(x)}{4\phi_1^2} + \frac{|P(x)|}{3\phi_2}$$

and

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \frac{|P(x)|}{3\phi_2} & , \ 0 \leq |t(\delta;x)| \leq \frac{1}{6\phi_2} \\ 2\,|P(x)|\,|t(\delta;x)| & , \quad |t(\delta;x)| \geq \frac{1}{6\phi_2} \end{cases}$$

Choosing $\alpha = 1$, $\tau = 1$, $q \to 1^-$ and $\phi_1 = \phi_2 = 1$ in Theorems 1 and 2, we have Corollary 4

Corollary 4 *Let* $f \in Q^{\mu}(1,1;x) = Q^{\mu}(x)$ *be of the form* (4). *Also assume that* $\delta \in \mathbb{R}$. *If* $\phi_1 = \phi_2 = 1$, *then*

$$|a_2| \leq \frac{|P(x)|\sqrt{|P(x)|}}{\sqrt{\left|P^2(x) + 8Q(x)\right|}},$$

$$|a_3| \leq \frac{P^2(x)}{4} + \frac{|P(x)|}{3}$$

and

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \frac{|P(x)|}{3} & , \ 0 \leq |t(\delta;x)| \leq \frac{1}{6} \\ 2\,|P(x)|\,|t(\delta;x)| & , \quad |t(\delta;x)| \geq \frac{1}{6}. \end{cases}$$

In particular, for $\delta = 1$, *we obtain*

$$\left|a_3 - a_2^2\right| \leq |\tau|\,\frac{|P(x)|}{(1+\alpha q+\alpha q^2)\phi_2},$$

and for $\delta = \tau = \alpha = 1$ *we obtain*

$$\left|a_3 - a_2^2\right| \leq \frac{|P(x)|}{3\phi_2}.$$

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