Univers. J. Math. Appl., 9 (3) (2026) 150-166 https://doi.org/10.32323/ujma.1918223

Universal Journal of Mathematics and Applications

ISSN (Online) 2619-9653
https://dergipark.org.tr/en/pub/ujma

Research Article

On Fractional Milne-Mercer Bounds for s-Convex Functions with Applications

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Article Info

Keywords: *Milne-Mercer type, s-convex function, Riemann-Liouville fractional integrals, Hölder's inequality, Power-mean inequality, Bounded function, Lipschitz function, q-digamma function, Modified Bessel function*

Received: *28 March 2026*
Accepted: *31 July 2026*
Available online: *2 August 2026*

Abstract

Fractional integral inequalities play a fundamental role in modern analysis and have significant applications in the qualitative study of fractional differential and integral equations. Motivated by the need for sharper estimates in fractional settings, this paper establishes several new Milne-Mercer type inequalities involving the Riemann-Liouville fractional integral operators. By employing the theory of s-convex functions, we derive a unified framework that extends and generalizes a variety of classical Mercer-type inequalities available in the literature. Furthermore, refined estimates are obtained for bounded functions and functions satisfying Lipschitz-type conditions. The proposed results not only recover numerous known inequalities as particular cases but also provide stronger bounds within the fractional calculus setting. To illustrate the applicability and effectiveness of the theoretical findings, graphical examples are presented. In addition, applications to quadrature formulas, q-digamma functions, and modified Bessel functions are discussed, demonstrating the broad utility of the developed inequalities. The results contribute to the ongoing development of fractional integral inequalities and offer new tools for the analysis of special functions and related problems in applied mathematics.

1. Introduction

Integral inequalities constitute a central component of modern mathematical analysis and arise naturally in a broad spectrum of problems, including approximation theory, operator theory, spectral analysis, statistics, and probability distribution modelling. Their relevance is particularly pronounced in the study of differential and integral equations, where such inequalities provide effective analytical bounds and stability criteria. In recent years, a substantial body of research has focused on extending classical integral inequalities to fractional frameworks, driven by the growing importance of fractional derivatives and integrals in modelling memory and hereditary dependent processes [1]. Within this context, convexity-based approaches have proved especially fruitful, leading to strengthened, sharpened, and generalized variants of numerous well-known inequalities, including the Hermite-Hadamard [2], Grüss, Simpson's, Jensen, Milne [3], Minkowski and power mean inequalities and reverse Hölder types. These advancements not only deepen the theoretical understanding of fractional operators but also offer robust numerical tools for analyzing complex mathematical models. Owing to their wide applicability in pure and applied sciences, the development of fractional integral inequalities along diverse classes of convex functions remains an active and rapidly evolving research direction.

Fractional analysis extends the framework of classical calculus by introducing operators of non-integer order, providing a versatile mathematical tool for modelling complex processes with memory and hereditary effects. Over the years, numerous formulations of fractional derivatives and integrals have been proposed, each equivalent in structure but differing in interpretation and physical significance. Among

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Cite as: A. Munir, H. Budak, A. Kashuri, S. I. Butt, *On fractional Milne-Mercer bounds for s-convex functions with applications*, Univers. J. Math. Appl., 9(3) (2026), 150-166.

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these, the Riemann-Liouville operator is recognized as one of the earliest and most fundamental fractional integral operators. The growing interest in fractional calculus is largely attributed to its broad applicability in numerical and physical sciences, fluid dynamics, biological and ecological modelling [4], and many other disciplines. The rapid development of this field has also been driven by advancements in convexity theory, particularly those rooted in Jensen's foundational ideas. These connections have fostered significant progress in functional analysis, stochastic processes, finite element methods, and optimization theory, further enriching the theoretical landscape of fractional calculus. As a result, fractional analysis has evolved into a unifying framework with deep links to quantum theory, orthogonal polynomials, and combinatorial structures. These motivations continue to inspire extensive research into new inequalities and analytical techniques within the fractional setting [5].

Let $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ and let $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ be a set of non-negative weights satisfying $\sum_{i=1}^{n} \theta_i = 1$. The classical Jensen inequality, as presented in [6], can be stated in the following form: If the function $\eth$ is convex on the interval $[\dot{a}, \ddot{a}]$, then

$$\eth\left(\sum_{i=1}^{n} \theta_i x_i\right) \leq \sum_{i=1}^{n} \theta_i \eth(x_i),$$

for all $x_i \in [\dot{a}, \ddot{a}]$, $\theta_i \in [0,1]$ and $(i = 1,2,\ldots,n)$. This inequality provides a foundation for deriving refined bounds for various distance measures in information theory (see [7]).

In 2003, Mercer introduced an important refinement of Jensen's inequality in [8], which is now widely known as the Jensen-Mercer inequality. The result can be stated as follows:

Theorem 1.1. *For a convex function $\eth : [\dot{a}, \ddot{a}] \to \mathbb{R}$, the following inequality holds for all $x_i \in [\dot{a}, \ddot{a}]$ $(i = 1,2,\ldots,n)$:*

$$\eth\left(\dot{a} + \ddot{a} - \sum_{i=1}^{n} \theta_i x_i\right) \leq \eth(\dot{a}) + \eth(\ddot{a}) - \sum_{i=1}^{n} \theta_i \eth(x_i),$$

where $\theta_i \in [0,1]\,(i = 1,2,\ldots,n)$ and $\sum_{i=1}^{n} \theta_i = 1$.

As a recent contribution to the theory of integral inequalities Hazma *et al.* [9] established novel error bounds for Milne formula-type inequalities within the framework of quantum calculus and demonstrated their effectiveness through computational analysis and practical applications. Subsequently, Singh *et al*. [10] established new Milne-type inequalities for strongly convex functions by employing Riemann-Liouville fractional integrals together with Hölder's and power-mean inequalities. Kian [11] further examined Jensen's inequality within the framework of superquadratic functions. In addition, reverse Mercer type operator inequalities, along with related results for superquadratic functions were developed in [12], while integral and integral mean variants were presented in [13]. For further recent developments on Jensen-Mercer type inequalities, readers may consult [14] and the references therein.

Moreover, in recent years the role of convex functions in the development of inequality theory has been extensively broadened and refined by numerous researchers. A commonly used formulation of convexity, along with its generalizations, is given as follows:

Definition 1.2. *[15] A function $\eth : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on I if, for any $\dot{a}, \ddot{a} \in I$ and any $\Bbbk_1 \in [0,1]$, the following inequality is satisfied:*

$$\eth(\Bbbk_1 \dot{a} + (1 - \Bbbk_1)\ddot{a}) \leq \Bbbk_1 \eth(\dot{a}) + (1 - \Bbbk_1)\eth(\ddot{a}).$$

Definition 1.3. *[16] A function $\eth : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense on I if:*

$$\eth(\Bbbk_1 \dot{a} + (1 - \Bbbk_1)\ddot{a}) \leq \Bbbk_1^s \eth(\dot{a}) + (1 - \Bbbk_1)^s \eth(\ddot{a}),$$

for all $\dot{a}, \ddot{a} \in I$ and $\Bbbk_1 \in [0,1]$ for some fixed $s \in (0,1]$.

Butt *et al.* [17] explored a Jensen-Mercer type inequality for s-convex functions in the sense of Breckner, for $s > 0$, which can be stated as follows:

Theorem 1.4. *Let $\theta_1, \theta_2, \ldots, \theta_n$ be positive real numbers with $n \geq 2$, satisfying $\sum_{i=1}^{n} \theta_i = 1$ and $\sum_{i=1}^{n} \theta_i^s \leq 1$. If $\eth$ is a real-valued Breckner s-convex function on $[\dot{a}, \ddot{a}] \subset \mathbb{R}^+$, then for any finite positive increasing sequence $\{x_i\}_i^n \in [\dot{a}, \ddot{a}]$, we have*

$$\eth\left(\dot{a} + \ddot{a} - \sum_{i=1}^{n} \theta_i x_i\right) \leq \eth(\dot{a}) + \eth(\ddot{a}) - \sum_{i=1}^{n} \theta_i^s \eth(x_i).$$

In this context, Milne's open-type formula plays a role analogous to Simpson's closed type formula within the Newton-Cotes family of numerical integration methods. Although structurally distinct, both formulas operate under similar assumptions, and their parallel behavior has motivated further investigation into integral inequalities associated with convex functions. This study contributes to this growing line of research by exploring new extensions and applications inspired by these classical quadrature identities.

Theorem 1.5. *[18] Suppose that $\eth : [\dot{a}, \ddot{a}] \to \mathbb{R}$ is a four-times continuously differentiable function on $(\dot{a}, \ddot{a})$, and let $\left\|\eth^{(4)}\right\|_\infty := \sup_{x \in (\dot{a}, \ddot{a})} \left|\eth^{(4)}(x)\right| dx < \infty$. Then, one has the following inequality:*

$$\left|\frac{1}{3}\left[2\eth(\dot{a}) - \eth\left(\frac{\dot{a} + \ddot{a}}{2}\right) + 2\eth(\ddot{a})\right] - \frac{1}{\ddot{a} - \dot{a}} \int_{\dot{a}}^{\ddot{a}} \eth(u)\, du\right| \leq \frac{7(\ddot{a} - \dot{a})^4}{23040} \left\|\eth^{(4)}\right\|_\infty.$$

Several researchers have focused on the development of Milne-type inequalities for various classes of functions. In particular, a number of studies have examined these inequalities within the framework of convex mappings, as convexity provides a powerful and reliable approach for addressing numerous problems arising across pure and applied sciences. For recent advancements involving Milne-type inequalities established through different notions of convexity and fractional operators, readers are referred to [19, 20] and the references therein.

Definition 1.6. *[21] Let $\eth \in L[\dot{a},\ddot{a}]$. The Riemann-Liouville fractional integral operators of order $\alpha > 0$, in both left and right-sided forms, are introduced by:*

$$J^{\alpha}_{\dot{a}^{+}}\eth(x) = \frac{1}{\Gamma(\alpha)}\int_{\dot{a}}^{x}(x-\Bbbk_1)^{\alpha-1}\eth(\Bbbk_1)\,d\Bbbk_1, \quad x > \dot{a},$$

and

$$J^{\alpha}_{\ddot{a}^{-}}\eth(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\ddot{a}}(\Bbbk_1-x)^{\alpha-1}\eth(\Bbbk_1)\,d\Bbbk_1, \; x < \ddot{a}.$$

Here, $\Gamma(\alpha)$ is the Gamma function.

Benaissa *et al.* [22] extended the classical Milne-Mercer-type inequalities for h-convex functions by employing a generalized fractional integral framework. Building on this development, Benaissa *et al.* [23] further extended these results by establishing Milne-type inequalities for h-convex functions via conformable fractional integral operators. Çelik *et al.* [24] further advanced this line of research by establishing generalized Milne-type inequalities through new conformable fractional integral operators. Arslan *et al.* [25] employed the Atangana-Baleanu fractional integral operator to establish new Milne-type inequalities for convex functions. In another line of work, Benaissa *et al.* [14] established error bounds for Milne-Mercer-type inequalities involving differentiable s-convex functions. Mohsin *et al.* [26] established novel Milne-Mercer-type inequalities in the framework of quantum calculus.

The proposed results not only will recover numerous known inequalities as particular cases but also will provide stronger bounds within the fractional calculus setting. To illustrate the applicability and effectiveness of the theoretical findings, graphical examples will be present. In addition, applications to quadrature formulas, q-digamma functions, and modified Bessel functions will be discussed, demonstrating the broad utility of the developed inequalities. The results contribute to the ongoing development of fractional integral inequalities and offer new tools for the analysis of special functions and related problems in applied mathematics.

The remainder of the paper is organized as follows. In Section 2, a novel integral identity utilizing the Riemann-Liouville fractional integral operators is established. Based on this identity, we will develop new Milne-Mercer-type inequalities for s-convex functions. Moreover, fractional Milne-Mercer-type inequality for bounded functions and some special cases will be given. Furthermore, some fractional Milne-Mercer-type inequality for Lipschitz-type functions will be discussed. In Section 3, some applications to Milne's quadrature formula, q-digamma functions, and modified Bessel functions will be explored. In Section 4, to highlight the effectiveness of the proposed results, graphical demonstrations will be obtained. In Section 5, we provide a comparative analysis. In Section 6, some conclusions and future research will be given.

2. Fractional Milne-Mercer-Type Inequalities via s-Convexity

In this section, we investigate a variety of Milne-Mercer type inequalities using the Riemann-Liouville fractional integral operator for differentiable functions which are s-convex. To facilitate this study, we first establish a novel Riemann-Liouville integral identity, which serves as a foundational tool for deriving the subsequent refined inequalities.

Lemma 2.1. *Let $\eth : [\dot{a},\ddot{a}] \to \mathbb{R}$ be a differentiable mapping such that $\eth' \in L[\dot{a},\ddot{a}]$, where $\dot{a} < \ddot{a}$. For any $\tau, \upsilon \in [\dot{a},\ddot{a}]$, then the following Milne-Mercer type equality holds:*

$$\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau) - \eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right) + 2\eth(\dot{a}+\ddot{a}-\upsilon)\right]$$

$$-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha}}\left[J^{\alpha}_{(\dot{a}+\ddot{a}-\upsilon)^{+}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right) + J^{\alpha}_{(\dot{a}+\ddot{a}-\tau)^{-}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right]$$

$$= \frac{\upsilon-\tau}{4}\int_0^1\left(\Bbbk_1^{\alpha}+\frac{1}{3}\right)\left[\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1+\Bbbk_1}{2}\tau+\frac{1-\Bbbk_1}{2}\upsilon\right)\right) - \eth'\left(\dot{a}+\ddot{a}-\left(\frac{1-\Bbbk_1}{2}\tau+\frac{1+\Bbbk_1}{2}\upsilon\right)\right)d\Bbbk_1\right].$$

Proof. Let

$$\begin{aligned} I &= \int_0^1\left(\Bbbk_1^{\alpha}+\frac{1}{3}\right)\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1+\Bbbk_1}{2}\tau+\frac{1-\Bbbk_1}{2}\upsilon\right)\right)d\Bbbk_1 - \int_0^1\left(\Bbbk_1^{\alpha}+\frac{1}{3}\right)\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1-\Bbbk_1}{2}\tau+\frac{1+\Bbbk_1}{2}\upsilon\right)\right)d\Bbbk_1 \\ &= I_2 - I_1. \end{aligned}$$

Applying integration by parts gives

$$I_1 = \int_0^1 \left(\mathbb{k}_1^{\alpha} + \frac{1}{3}\right) \eth'\left(\dot{a}+\ddot{a} - \left(\frac{1-\mathbb{k}_1}{2}\tau + \frac{1+\mathbb{k}_1}{2}\upsilon\right)\right) d\mathbb{k}_1$$

$$= \frac{-2}{\upsilon-\tau}\left(\mathbb{k}_1 + \frac{1}{3}\right)\eth\left(\dot{a}+\ddot{a} - \left(\frac{1-\mathbb{k}_1}{2}\tau + \frac{1+\mathbb{k}_1}{2}\upsilon\right)\right)\Big|_0^1 + \frac{2\alpha}{\upsilon-\tau}\int_0^1 \mathbb{k}_1^{\alpha-1}\eth\left(\dot{a}+\ddot{a} - \left(\frac{1-\mathbb{k}_1}{2}\tau + \frac{1+\mathbb{k}_1}{2}\upsilon\right)\right) d\mathbb{k}_1$$

$$= \frac{-8}{3(\upsilon-\tau)}\eth(\dot{a}+\ddot{a}-\upsilon) + \frac{2}{3(\upsilon-\tau)}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right) + \frac{2^{\alpha+1}}{(\upsilon-\tau)^{\alpha+1}}\int_{\dot{a}+\ddot{a}-\upsilon}^{\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}} \eth\left(\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)-u\right)^{\alpha-1} du$$

$$= \frac{-8}{3(\upsilon-\tau)}\eth(\dot{a}+\ddot{a}-\upsilon) + \frac{2}{3(\upsilon-\tau)}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right) + \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha+1}} J^{\alpha}_{(\dot{a}+\ddot{a}-\upsilon)^+}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right). \tag{2.1}$$

In addition, we obtain

$$I_2 = \int_0^1 \left(\mathbb{k}_1^{\alpha} + \frac{1}{3}\right) \eth'\left(\dot{a}+\ddot{a} - \left(\frac{1+\mathbb{k}_1}{2}\tau + \frac{1-\mathbb{k}_1}{2}\upsilon\right)\right) d\mathbb{k}_1$$

$$= \frac{2}{\upsilon-\tau}\left(\mathbb{k}_1 + \frac{1}{3}\right)\eth\left(\dot{a}+\ddot{a} - \left(\frac{1+\mathbb{k}_1}{2}\tau + \frac{1-\mathbb{k}_1}{2}\upsilon\right)\right)\Big|_0^1 - \frac{2\alpha}{\upsilon-\tau}\int_0^1 \mathbb{k}_1^{\alpha-1}\eth\left(\dot{a}+\ddot{a} - \left(\frac{1+\mathbb{k}_1}{2}\tau + \frac{1-\mathbb{k}_1}{2}\upsilon\right)\right) d\mathbb{k}_1$$

$$= \frac{8}{3(\upsilon-\tau)}\eth(\dot{a}+\ddot{a}-\tau) - \frac{2}{3(\upsilon-\tau)}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right) - \frac{2^{\alpha+1}}{(\upsilon-\tau)^{\alpha+1}}\int_{\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}}^{\dot{a}+\ddot{a}-\tau} \eth\left(u-\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right)^{\alpha-1} du$$

$$\frac{8}{3(\upsilon-\tau)}\eth(\dot{a}+\ddot{a}-\tau) - \frac{2}{3(\upsilon-\tau)}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right) - \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha+1}} J^{\alpha}_{(\dot{a}+\ddot{a}-\tau)^-}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right). \tag{2.2}$$

Subtracting above equalities (2.2) and (2.1), then we have

$$I_2 - I_1 = \frac{8}{3(\upsilon-\tau)}\eth(\dot{a}+\ddot{a}-\tau) - \frac{2}{3(\upsilon-\tau)}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right) - \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha+1}} J^{\alpha}_{(\dot{a}+\ddot{a}-\tau)^-}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)$$

$$+ \frac{8}{3(\upsilon-\tau)}\eth(\dot{a}+\ddot{a}-\upsilon) - \frac{2}{3(\upsilon-\tau)}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right) - \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha+1}} J^{\alpha}_{(\dot{a}+\ddot{a}-\upsilon)^+}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)$$

$$= \frac{8}{3(\upsilon-\tau)}\eth(\dot{a}+\ddot{a}-\tau) + \frac{8}{3(\upsilon-\tau)}\eth(\dot{a}+\ddot{a}-\upsilon) - \frac{4}{3(\upsilon-\tau)}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)$$

$$- \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha+1}}\left[J^{\alpha}_{(\dot{a}+\ddot{a}-\upsilon)^+}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right) + J^{\alpha}_{(\dot{a}+\ddot{a}-\tau)^-}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right].$$

Finally, multiplying both sides by $\frac{\upsilon-\tau}{4}$, we obtain

$$\frac{\upsilon-\tau}{4}(I_2 - I_1) = \frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau) - \eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right) + 2\eth(\dot{a}+\ddot{a}-\upsilon)\right]$$

$$- \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha}}\left[J^{\alpha}_{(\dot{a}+\ddot{a}-\upsilon)^+}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right) + J^{\alpha}_{(\dot{a}+\ddot{a}-\tau)^-}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right].$$

Therefore, the proof of Lemma 2.1 is completed. □

Corollary 2.2. *From Lemma 2.1 with $\alpha = 1$, we obtain*

$$\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau) - \eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right) + 2\eth(\dot{a}+\ddot{a}-\upsilon)\right] - \frac{1}{\upsilon-\tau}\int_{\dot{a}+\ddot{a}-\upsilon}^{\dot{a}+\ddot{a}-\tau}\eth(u)\,du$$

$$= \frac{\upsilon-\tau}{4}\int_0^1 \left(\mathbb{k}_1 + \frac{1}{3}\right)\left[\eth'\left(\dot{a}+\ddot{a} - \left(\frac{1+\mathbb{k}_1}{2}\tau + \frac{1-\mathbb{k}_1}{2}\upsilon\right)\right) - \eth'\left(\dot{a}+\ddot{a} - \left(\frac{1-\mathbb{k}_1}{2}\tau + \frac{1+\mathbb{k}_1}{2}\upsilon\right)\right) d\mathbb{k}_1\right].$$

Theorem 2.3. *Let $\eth : [\dot{a},\ddot{a}] \to \mathbb{R}$ be a differentiable mapping such that $\eth' \in L[\dot{a},\ddot{a}]$. Suppose $|\eth'|$ is s-convex on $[\dot{a},\ddot{a}]$ for some fixed $s \in (0,1]$ with $\dot{a} < \ddot{a}$. For any $\tau, \upsilon \in [\dot{a},\ddot{a}]$, the following fractional Milne-Mercer type inequality holds:*

$$\left|\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth(\dot{a}+\ddot{a}-\upsilon)\right]\right.$$

$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha}}\left[J^{\alpha}_{(\dot{a}+\ddot{a}-\upsilon)^{+}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+J^{\alpha}_{(\dot{a}+\ddot{a}-\tau)^{-}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{\upsilon-\tau}{4}\left[\left(\frac{2}{\alpha+1}+\frac{2}{3}\right)\left(\left|\eth'(\dot{a})\right|+\left|\eth'(\ddot{a})\right|\right)-\left(\frac{2}{3(1+s)}+\frac{2^{-s}\Gamma(1+s)\Gamma(1+\alpha)}{\Gamma(s+\alpha+2)}\right.\right.$$

$$\left.\left.+2^{-s}\left(\frac{1}{\alpha+1}\right){}_2F_1(-s,\alpha+1;\alpha+2;-1)+\frac{2}{3(s+1)}\left(1-2^{-(s+1)}\right)\right)\left(\left|\eth'(\tau)\right|+\left|\eth'(\upsilon)\right|\right)\right],$$

where ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the well-known hypergeometric function.

Proof. Taking the absolute value in Lemma 2.1, yields

$$\left|\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth(\dot{a}+\ddot{a}-\upsilon)\right]\right.$$

$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha}}\left[J^{\alpha}_{(\dot{a}+\ddot{a}-\upsilon)^{+}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+J^{\alpha}_{(\dot{a}+\ddot{a}-\tau)^{-}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{\upsilon-\tau}{4}\left[\int_0^1\left(\mathbb{k}_1^{\alpha}+\frac{1}{3}\right)\left|\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1+\mathbb{k}_1}{2}\tau+\frac{1-\mathbb{k}_1}{2}\upsilon\right)\right)\right|d\mathbb{k}_1\right.$$

$$\left.+\int_0^1\left(\mathbb{k}_1^{\alpha}+\frac{1}{3}\right)\left|\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1-\mathbb{k}_1}{2}\tau+\frac{1+\mathbb{k}_1}{2}\upsilon\right)\right)\right|d\mathbb{k}_1\right].$$

Using the s-convexity of $|\eth'|$, we obtain

$$\left|\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth(\dot{a}+\ddot{a}-\upsilon)\right]\right.$$

$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha}}\left[J^{\alpha}_{(\dot{a}+\ddot{a}-\upsilon)^{+}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+J^{\alpha}_{(\dot{a}+\ddot{a}-\tau)^{-}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{\upsilon-\tau}{4}\left[\int_0^1\left(\mathbb{k}_1^{\alpha}+\frac{1}{3}\right)\left|\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1+\mathbb{k}_1}{2}\tau+\frac{1-\mathbb{k}_1}{2}\upsilon\right)\right)\right|d\mathbb{k}_1\right.$$

$$\left.+\int_0^1\left(\mathbb{k}_1^{\alpha}+\frac{1}{3}\right)\left|\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1-\mathbb{k}_1}{2}\tau+\frac{1+\mathbb{k}_1}{2}\upsilon\right)\right)\right|d\mathbb{k}_1\right]$$

$$\leq \frac{\upsilon-\tau}{4}\left[\int_0^1\left(\mathbb{k}_1^{\alpha}+\frac{1}{3}\right)\left(\left(\left|\eth'(\dot{a})\right|+\left|\eth'(\ddot{a})\right|\right)-\left(\left(\frac{1+\mathbb{k}_1}{2}\right)^{s}\left|\eth'(\tau)\right|+\left(\frac{1-\mathbb{k}_1}{2}\right)^{s}\left|\eth'(\upsilon)\right|\right)\right)d\mathbb{k}_1\right.$$

$$\left.+\int_0^1\left(\mathbb{k}_1^{\alpha}+\frac{1}{3}\right)\left(\left(\left|\eth'(\dot{a})\right|+\left|\eth'(\ddot{a})\right|\right)-\left(\left(\frac{1-\mathbb{k}_1}{2}\right)^{s}\left|\eth'(\tau)\right|+\left(\frac{1+\mathbb{k}_1}{2}\right)^{s}\left|\eth'(\upsilon)\right|\right)\right)d\mathbb{k}_1\right]$$

$$=\frac{\upsilon-\tau}{4}\left[\left(\frac{2}{\alpha+1}+\frac{2}{3}\right)\left(\left|\eth'(\dot{a})\right|+\left|\eth'(\ddot{a})\right|\right)-\left(\frac{2}{3(1+s)}+\frac{2^{-s}\Gamma(1+s)\Gamma(1+\alpha)}{\Gamma(s+\alpha+2)}\right.\right.$$

$$\left.\left.+2^{-s}\left(\frac{1}{\alpha+1}\right){}_2F_1(-s,\alpha+1;\alpha+2;-1)+\frac{2}{3(s+1)}\left(1-2^{-(s+1)}\right)\right)\left(\left|\eth'(\tau)\right|+\left|\eth'(\upsilon)\right|\right)\right].$$

Therefore, the proof of Theorem 2.3 is completed. □

Corollary 2.4. *By setting $\alpha = 1$ in Theorem 2.3, we obtain*

$$\left|\frac{1}{3}\left[2\eth\left(\dot{a}+\ddot{a}-\tau\right)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth\left(\dot{a}+\ddot{a}-\upsilon\right)\right]-\int_{\dot{a}+\ddot{a}-\upsilon}^{\dot{a}+\ddot{a}-\tau}\eth\left(u\right)du\right|$$

$$\leq\frac{\upsilon-\tau}{4}\left[\frac{5}{3}\left(\left|\eth'\left(\dot{a}\right)\right|+\left|\eth'\left(\ddot{a}\right)\right|\right)-\left(\frac{2}{3\left(1+s\right)}+\frac{2^{-s}\Gamma\left(1+s\right)}{\Gamma\left(s+3\right)}\right.\right.$$

$$\left.\left.+2^{-s}\left(\frac{1}{2}\right){}_2F_1\left(-s,2;3;-1\right)+\frac{2}{3\left(s+1\right)}\left(1-2^{-\left(s+1\right)}\right)\right)\left(\left|\eth'\left(\tau\right)\right|+\left|\eth'\left(\upsilon\right)\right|\right)\right].$$

Corollary 2.5. *If $s = 1$ in Corollary 2.4, it follows that*

$$\left|\frac{1}{3}\left[2\eth\left(\dot{a}+\ddot{a}-\tau\right)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth\left(\dot{a}+\ddot{a}-\upsilon\right)\right]-\frac{1}{\upsilon-\tau}\int_{\dot{a}+\ddot{a}-\upsilon}^{\dot{a}+\ddot{a}-\tau}\eth\left(u\right)du\right|$$

$$\leq\frac{\upsilon-\tau}{4}\left[\frac{5}{3}\left(\left|\eth'\left(\dot{a}\right)\right|+\left|\eth'\left(\ddot{a}\right)\right|\right)-\frac{5}{6}\left(\left|\eth'\left(\tau\right)\right|+\left|\eth'\left(\upsilon\right)\right|\right)\right].$$

Remark 2.6. *Taking $\tau = \dot{a}$ and $\upsilon = \ddot{a}$ in Corollary 2.5, it follows that*

$$\left|\frac{1}{3}\left[2\eth\left(\dot{a}\right)-\eth\left(\frac{\dot{a}+\ddot{a}}{2}\right)+2\eth\left(\ddot{a}\right)\right]-\frac{1}{\ddot{a}-\dot{a}}\int_{\dot{a}}^{\ddot{a}}\eth\left(u\right)du\right|\leq\frac{5\left(\ddot{a}-\dot{a}\right)}{24}\left(\left|\eth'\left(\dot{a}\right)\right|+\left|\eth'\left(\ddot{a}\right)\right|\right),$$

The above result was derived by Djenaoui et al. [27, Corollary 2.4].

Theorem 2.7. *Under the conditions stated in Theorem 2.3 holds. If $|\eth'|^q$ for $q > 1$ and $\frac{1}{p}+\frac{1}{q}=1$ is s-convex on $[\dot{a},\ddot{a}]$ for some fixed $s \in (0,1]$, then the following fractional Milne-Mercer type inequality holds:*

$$\left|\frac{1}{3}\left[2\eth\left(\dot{a}+\ddot{a}-\tau\right)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth\left(\dot{a}+\ddot{a}-\upsilon\right)\right]\right.$$

$$\left.-\frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(\upsilon-\tau\right)^{\alpha}}\left[J^{\alpha}_{\left(\dot{a}+\ddot{a}-\upsilon\right)^{+}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+J^{\alpha}_{\left(\dot{a}+\ddot{a}-\tau\right)^{-}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right]\right|$$

$$\leq\frac{\upsilon-\tau}{4}\left(\int_0^1\left(\Bbbk_1^{\alpha}+\frac{1}{3}\right)^p d\Bbbk_1\right)^{\frac{1}{p}}\left[\left(\left(\left|\eth'\left(\dot{a}\right)\right|^q+\left|\eth'\left(\ddot{a}\right)\right|^q\right)-\left(\frac{2-2^{-s}}{\left(s+1\right)}\left|\eth'\left(\tau\right)\right|^q+\frac{2^{-s}}{\left(s+1\right)}\left|\eth'\left(\upsilon\right)\right|^q\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left(\left|\eth'\left(\dot{a}\right)\right|^q+\left|\eth'\left(\ddot{a}\right)\right|^q\right)-\left(\frac{2^{-s}}{\left(s+1\right)}\left|\eth'\left(\tau\right)\right|^q+\frac{2-2^{-s}}{\left(s+1\right)}\left|\eth'\left(\upsilon\right)\right|^q\right)\right)^{\frac{1}{q}}\right].$$

Proof. Applying the absolute value to Lemma 2.1, we obtain

$$\left|\frac{1}{3}\left[2\eth\left(\dot{a}+\ddot{a}-\tau\right)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth\left(\dot{a}+\ddot{a}-\upsilon\right)\right]\right.$$

$$\left.-\frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(\upsilon-\tau\right)^{\alpha}}\left[J^{\alpha}_{\left(\dot{a}+\ddot{a}-\upsilon\right)^{+}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+J^{\alpha}_{\left(\dot{a}+\ddot{a}-\tau\right)^{-}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right]\right|$$

$$\leq\frac{\upsilon-\tau}{4}\left[\int_0^1\left(\Bbbk_1^{\alpha}+\frac{1}{3}\right)\left|\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1+\Bbbk_1}{2}\tau+\frac{1-\Bbbk_1}{2}\upsilon\right)\right)\right|d\Bbbk_1\right.$$

$$\left.+\int_0^1\left(\Bbbk_1^{\alpha}+\frac{1}{3}\right)\left|\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1-\Bbbk_1}{2}\tau+\frac{1+\Bbbk_1}{2}\upsilon\right)\right)\right|d\Bbbk_1\right]. \tag{2.3}$$

Applying Hölder's inequality in (2.3) and utilizing the s-convexity of $|\eth'|^q$, we obtain

$$\left|\frac{1}{3}\left[2\eth\left(\dot{a}+\ddot{a}-\tau\right)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth\left(\dot{a}+\ddot{a}-\upsilon\right)\right]\right.$$

$$\left.-\frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(\upsilon-\tau\right)^{\alpha}}\left[J^{\alpha}_{\left(\dot{a}+\ddot{a}-\upsilon\right)^{+}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+J^{\alpha}_{\left(\dot{a}+\ddot{a}-\tau\right)^{-}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{\upsilon-\tau}{4}\left[\int_0^1\left(\Bbbk_1^{\alpha}+\frac{1}{3}\right)\left|\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1+\Bbbk_1}{2}\tau+\frac{1-\Bbbk_1}{2}\upsilon\right)\right)\right|d\Bbbk_1\right.$$

$$\left.+\int_0^1\left(\Bbbk_1^{\alpha}+\frac{1}{3}\right)\left|\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1-\Bbbk_1}{2}\tau+\frac{1+\Bbbk_1}{2}\upsilon\right)\right)\right|d\Bbbk_1\right]$$

$$\leq \frac{\upsilon-\tau}{4}\left(\int_0^1\left(\Bbbk_1^{\alpha}+\frac{1}{3}\right)^p d\Bbbk_1\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1+\Bbbk_1}{2}\tau+\frac{1-\Bbbk_1}{2}\upsilon\right)\right)\right|^q d\Bbbk_1\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left|\left(\dot{a}+\ddot{a}-\left(\frac{1-\Bbbk_1}{2}\tau+\frac{1+\Bbbk_1}{2}\upsilon\right)\right)\right|^q d\Bbbk_1\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{\upsilon-\tau}{4}\left(\int_0^1\left(\Bbbk_1^{\alpha}+\frac{1}{3}\right)^p d\Bbbk_1\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_0^1\left(\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)-\left(\left(\frac{1+\Bbbk_1}{2}\right)^s\left|\eth'(\tau)\right|^q+\left(\frac{1-\Bbbk_1}{2}\right)^s\left|\eth'(\upsilon)\right|^q\right)\right)d\Bbbk_1\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left(\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)-\left(\left(\frac{1-\Bbbk_1}{2}\right)^s\left|\eth'(\tau)\right|^q+\left(\frac{1+\Bbbk_1}{2}\right)^s\left|\eth'(\upsilon)\right|^q\right)\right)d\Bbbk_1\right)^{\frac{1}{q}}\right]$$

$$=\frac{\upsilon-\tau}{4}\left(\int_0^1\left(\Bbbk_1^{\alpha}+\frac{1}{3}\right)^p d\Bbbk_1\right)^{\frac{1}{p}}\left[\left(\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)-\left(\frac{2-2^{-s}}{(s+1)}\left|\eth'(\tau)\right|^q+\frac{2^{-s}}{(s+1)}\left|\eth'(\upsilon)\right|^q\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)-\left(\frac{2^{-s}}{(s+1)}\left|\eth'(\tau)\right|^q+\frac{2-2^{-s}}{(s+1)}\left|\eth'(\upsilon)\right|^q\right)\right)^{\frac{1}{q}}\right].$$

Thus, Theorem 2.7 is established. □

Corollary 2.8. *By setting $\alpha=1$ in Theorem 2.7, we obtain*

$$\left|\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth(\dot{a}+\ddot{a}-\upsilon)\right]-\frac{1}{\upsilon-\tau}\int_{\dot{a}+\ddot{a}-\upsilon}^{\dot{a}+\ddot{a}-\tau}\eth(u)\,du\right|$$

$$\leq \frac{\upsilon-\tau}{4}\left(\frac{3^{-1-p}\left(4^{p+1}-1\right)}{(p+1)}\right)^{\frac{1}{p}}\left[\left(\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)-\left(\frac{2-2^{-s}}{(s+1)}\left|\eth'(\tau)\right|^q+\frac{2^{-s}}{(s+1)}\left|\eth'(\upsilon)\right|^q\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)-\left(\frac{2^{-s}}{(s+1)}\left|\eth'(\tau)\right|^q+\frac{2-2^{-s}}{(s+1)}\left|\eth'(\upsilon)\right|^q\right)\right)^{\frac{1}{q}}\right].$$

Corollary 2.9. *By setting $s=1$ in Corollary 2.8, we obtain*

$$\left|\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth(\dot{a}+\ddot{a}-\upsilon)\right]-\frac{1}{\upsilon-\tau}\int_{\dot{a}+\ddot{a}-\upsilon}^{\dot{a}+\ddot{a}-\tau}\eth(u)\,du\right|$$

$$\leq \frac{\upsilon-\tau}{4}\left(\frac{3^{-1-p}\left(4^{p+1}-1\right)}{(p+1)}\right)^{\frac{1}{p}}\left[\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q-\left(\frac{3\left|\eth'(\tau)\right|^q+\left|\eth'(\upsilon)\right|^q}{4}\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q-\left(\frac{\left|\eth'(\tau)\right|^q+3\left|\eth'(\upsilon)\right|^q}{4}\right)\right)^{\frac{1}{q}}\right].$$

Theorem 2.10. *Under the hypotheses of Theorem 2.3 holds. If $|\eth'|^q$ for $q \geq 1$ is s-convex on $[\dot{a}, \ddot{a}]$, then the following fractional Milne-Mercer type inequality holds:*

$$\left|\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth(\dot{a}+\ddot{a}-\upsilon)\right]\right.$$

$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha}}\left[J^{\alpha}_{(\dot{a}+\ddot{a}-\upsilon)^{+}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+J^{\alpha}_{(\dot{a}+\ddot{a}-\tau)^{-}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{\upsilon-\tau}{4}\left(\frac{\alpha+4}{3(\alpha+1)}\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{3+(\alpha+1)}{3(\alpha+1)}\right)\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)-\left(\left(\frac{2-2^{-s}}{3+3s}+\frac{2^{-s}{}_2F_1(-s,\alpha+1;\alpha+2;-1)}{\alpha+1}\right)\left|\eth'(\tau)\right|^q\right.\right.$$

$$\left.\left.+\left(-\frac{1}{3}\times 2^{-s}+\left(\frac{1}{s+1}+\frac{3\Gamma(1+s)\Gamma(1+\alpha)}{\Gamma(s+\alpha+2)}\right)\right)\left|\eth'(\upsilon)\right|^q\right)\right)^{\frac{1}{q}}+\left(\left(\frac{3+(\alpha+1)}{3(\alpha+1)}\right)\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)\right.$$

$$\left.-\left(\left(-\frac{1}{3}\times 2^{-s}+\left(\frac{1}{s+1}+\frac{3\Gamma(1+s)\Gamma(1+\alpha)}{\Gamma(s+\alpha+2)}\right)\right)\left|\eth'(\tau)\right|^q+\left(\left(\frac{2-2^{-s}}{3+3s}+\frac{2^{-s}{}_2F_1(-s,\alpha+1;\alpha+2;-1)}{\alpha+1}\right)\right)\left|\eth'(\upsilon)\right|^q\right)\right)^{\frac{1}{q}}\right].$$

Proof. Taking the absolute value in Lemma 2.1 applying the power-mean inequality and using the s-convexity of $|\eth'|^q$, we obtain

$$\left|\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth(\dot{a}+\ddot{a}-\upsilon)\right]\right.$$

$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha}}\left[J^{\alpha}_{(\dot{a}+\ddot{a}-\upsilon)^{+}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+J^{\alpha}_{(\dot{a}+\ddot{a}-\tau)^{-}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{\upsilon-\tau}{4}\left[\int_0^1\left(\mathbb{k}_1^{\alpha}+\frac{1}{3}\right)\left|\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1+\mathbb{k}_1}{2}\tau+\frac{1-\mathbb{k}_1}{2}\upsilon\right)\right)\right|d\mathbb{k}_1+\int_0^1\left(\mathbb{k}_1^{\alpha}+\frac{1}{3}\right)\left|\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1-\mathbb{k}_1}{2}\tau+\frac{1+\mathbb{k}_1}{2}\upsilon\right)\right)\right|d\mathbb{k}_1\right]$$

$$\leq \frac{\upsilon-\tau}{4}\left(\int_0^1\left(\mathbb{k}_1^{\alpha}+\frac{1}{3}\right)d\mathbb{k}_1\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^1\left(\mathbb{k}_1^{\alpha}+\frac{1}{3}\right)\left|\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1+\mathbb{k}_1}{2}\tau+\frac{1-\mathbb{k}_1}{2}\upsilon\right)\right)\right|^q d\mathbb{k}_1\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left(\mathbb{k}_1^{\alpha}+\frac{1}{3}\right)\left|\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1+\mathbb{k}_1}{2}\tau+\frac{1-\mathbb{k}_1}{2}\upsilon\right)\right)\right|^q d\mathbb{k}_1\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{\upsilon-\tau}{4}\left(\int_0^1\left(\mathbb{k}_1^{\alpha}+\frac{1}{3}\right)d\mathbb{k}_1\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^1\left(\mathbb{k}_1^{\alpha}+\frac{1}{3}\right)\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)-\left(\left(\frac{1+\mathbb{k}_1}{2}\right)^s\left|\eth'(\tau)\right|^q+\left(\frac{1-\mathbb{k}_1}{2}\right)^s\left|\eth'(\upsilon)\right|^q\right)d\mathbb{k}_1\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left(\mathbb{k}_1^{\alpha}+\frac{1}{3}\right)\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)-\left(\left(\frac{1-\mathbb{k}_1}{2}\right)^s\left|\eth'(\tau)\right|^q+\left(\frac{1+\mathbb{k}_1}{2}\right)^s\left|\eth'(\upsilon)\right|^q\right)d\mathbb{k}_1\right)^{\frac{1}{q}}\right]$$

$$=\frac{\upsilon-\tau}{4}\left(\frac{\alpha+4}{3(\alpha+1)}\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{3+(\alpha+1)}{3(\alpha+1)}\right)\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)\right.\right.$$

$$\left.\left.-\left(\left(\frac{2-2^{-s}}{3+3s}+\frac{2^{-s}{}_2F_1(-s,\alpha+1;\alpha+2;-1)}{\alpha+1}\right)\left|\eth'(\tau)\right|^q+\left(-\frac{1}{3}\times 2^{-s}+\left(\frac{1}{s+1}+\frac{3\Gamma(1+s)\Gamma(1+\alpha)}{\Gamma(s+\alpha+2)}\right)\right)\left|\eth'(\upsilon)\right|^q\right)\right)^{\frac{1}{q}}$$

$$+\left(\left(\frac{3+(\alpha+1)}{3(\alpha+1)}\right)\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)-\left(\left(-\frac{1}{3}\times 2^{-s}+\left(\frac{1}{s+1}+\frac{3\Gamma(1+s)\Gamma(1+\alpha)}{\Gamma(s+\alpha+2)}\right)\right)\left|\eth'(\tau)\right|^q\right.$$
$$\left.+\left(\left(\frac{2-2^{-s}}{3+3s}+\frac{2^{-s}{}_2F_1(-s,\alpha+1;\alpha+2;-1)}{\alpha+1}\right)\right)\left|\eth'(\upsilon)\right|^q\right)\Bigg)^{\frac{1}{q}}\Bigg].$$

Hence, the proof of Theorem 2.10 is concluded. □

Corollary 2.11. *By setting $\alpha=1$ in Theorem 2.10, we obtain*

$$\left|\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth(\dot{a}+\ddot{a}-\upsilon)\right]-\frac{1}{\upsilon-\tau}\int_{\dot{a}+\ddot{a}-\upsilon}^{\dot{a}+\ddot{a}-\tau}\eth(u)\,du\right|$$

$$\leq\frac{\upsilon-\tau}{4}\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\Bigg[\left(\left(\frac{5}{6}\right)\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)\right.$$

$$-\left(\left(\frac{2-2^{-s}}{3+3s}+\frac{2^{-s}{}_2F_1(-s,2;3;-1)}{2}\right)\left|\eth'(\tau)\right|^q+\left(-\frac{1}{3}\times 2^{-s}+\left(\frac{1}{s+1}+\frac{3\Gamma(1+s)}{\Gamma(s+3)}\right)\right)\left|\eth'(\upsilon)\right|^q\right)\Bigg)^{\frac{1}{q}}$$

$$+\left(\left(\frac{5}{6}\right)\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)-\left(\left(-\frac{1}{3}\times 2^{-s}+\left(\frac{1}{s+1}+\frac{3\Gamma(1+s)}{\Gamma(s+3)}\right)\right)\left|\eth'(\tau)\right|^q\right.$$

$$\left.+\left(\left(\frac{2-2^{-s}}{3+3s}+\frac{2^{-s}{}_2F_1(-s,2;3;-1)}{2}\right)\right)\left|\eth'(\upsilon)\right|^q\right)\Bigg)^{\frac{1}{q}}\Bigg].$$

Corollary 2.12. *If we considered $s=1$ in Corollary 2.11, then it follows that*

$$\left|\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth(\dot{a}+\ddot{a}-\upsilon)\right]-\frac{1}{\upsilon-\tau}\int_{\dot{a}+\ddot{a}-\upsilon}^{\dot{a}+\ddot{a}-\tau}\eth(u)\,du\right|$$

$$\leq\frac{\upsilon-\tau}{4}\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\Bigg[\left(\frac{5}{6}\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)-\left(\frac{\left|\eth'(\tau)\right|^q+4\left|\eth'(\upsilon)\right|^q}{6}\right)\right)^{\frac{1}{q}}$$

$$+\left(\frac{5}{6}\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)-\left(\frac{4\left|\eth'(\tau)\right|^q+\left|\eth'(\upsilon)\right|^q}{6}\right)\right)^{\frac{1}{q}}\Bigg].$$

Theorem 2.13. *Let $\eth$ be introduced in Theorem 2.3 holds. Suppose there exist real constants h and H for which $h\leq\eth'(\Bbbk_1)\leq H$ for all $\Bbbk_1\in[\dot{a},\ddot{a}]$, one obtains the following fractional Milne-Mercer type inequality:*

$$\left|\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth(\dot{a}+\ddot{a}-\upsilon)\right]\right.$$

$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha}}\left[J^{\alpha}_{(\dot{a}+\ddot{a}-\upsilon)^+}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+J^{\alpha}_{(\dot{a}+\ddot{a}-\tau)^-}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right]\right|$$

$$\leq\frac{\upsilon-\tau}{4}\left(\frac{3+(\alpha+1)}{3(\alpha+1)}\right)(H-h).$$

Proof. Applying Lemma 2.1, we obtain

$$\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth(\dot{a}+\ddot{a}-\upsilon)\right]$$

$$-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha}}\left[J^{\alpha}_{(\dot{a}+\ddot{a}-\upsilon)^+}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+J^{\alpha}_{(\dot{a}+\ddot{a}-\tau)^-}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right]$$

$$=\frac{\upsilon-\tau}{4}\int_0^1\left(\Bbbk_1^{\alpha}+\frac{1}{3}\right)\left[\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1+\Bbbk_1}{2}\tau+\frac{1-\Bbbk_1}{2}\upsilon\right)\right)-\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1-\Bbbk_1}{2}\tau+\frac{1+\Bbbk_1}{2}\upsilon\right)\right)d\Bbbk_1\right]$$

$$= \frac{\upsilon-\tau}{4}\left[\int_0^1 \left(\Bbbk_1^{\alpha}+\frac{1}{3}\right)\left\{\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1+\Bbbk_1}{2}\tau+\frac{1-\Bbbk_1}{2}\upsilon\right)\right)-\frac{h+H}{2}\right\}d\Bbbk_1\right.$$

$$\left.-\int_0^1 \left(\Bbbk_1^{\alpha}+\frac{1}{3}\right)\left\{\frac{h+H}{2}-\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1-\Bbbk_1}{2}\tau+\frac{1+\Bbbk_1}{2}\upsilon\right)\right)\right\}d\Bbbk_1\right]. \tag{2.4}$$

By considering the absolute value of (2.4), we obtain

$$\left|\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth(\dot{a}+\ddot{a}-\upsilon)\right]\right.$$

$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha}}\left[J^{\alpha}_{(\dot{a}+\ddot{a}-\upsilon)^{+}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+J^{\alpha}_{(\dot{a}+\ddot{a}-\tau)^{-}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{\upsilon-\tau}{4}\left[\int_0^1 \left(\Bbbk_1^{\alpha}+\frac{1}{3}\right)\left|\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1+\Bbbk_1}{2}\tau+\frac{1-\Bbbk_1}{2}\upsilon\right)\right)-\frac{h+H}{2}\right|d\Bbbk_1\right.$$

$$\left.+\int_0^1 \left(\Bbbk_1^{\alpha}+\frac{1}{3}\right)\left|\frac{h+H}{2}-\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1-\Bbbk_1}{2}\tau+\frac{1+\Bbbk_1}{2}\upsilon\right)\right)\right|d\Bbbk_1\right].$$

From $h \leq \eth'(\Bbbk_1) \leq H$ for all $\Bbbk_1 \in [\dot{a},\ddot{a}]$, we get

$$\left|\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1+\Bbbk_1}{2}\tau+\frac{1-\Bbbk_1}{2}\upsilon\right)\right)-\frac{h+H}{2}\right| \leq \frac{H-h}{2}, \tag{2.5}$$

and

$$\left|\frac{h+H}{2}-\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1-\Bbbk_1}{2}\tau+\frac{1+\Bbbk_1}{2}\upsilon\right)\right)\right| \leq \frac{H-h}{2}. \tag{2.6}$$

By combining the relations given in (2.5) and (2.6), we obtain the following result

$$\left|\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth(\dot{a}+\ddot{a}-\upsilon)\right]\right.$$

$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha}}\left[J^{\alpha}_{(\dot{a}+\ddot{a}-\upsilon)^{+}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+J^{\alpha}_{(\dot{a}+\ddot{a}-\tau)^{-}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{\upsilon-\tau}{4}(H-h)\int_0^1 \left(\Bbbk_1^{\alpha}+\frac{1}{3}\right)d\Bbbk_1$$

$$= \frac{\upsilon-\tau}{4}\left(\frac{3+(\alpha+1)}{3(\alpha+1)}\right)(H-h).$$

This completes the proof of Theorem 2.13. □

Corollary 2.14. *By setting $\alpha = 1$ in Theorem 2.13, we obtain*

$$\left|\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth(\dot{a}+\ddot{a}-\upsilon)\right]-\frac{1}{\upsilon-\tau}\int_{\dot{a}+\ddot{a}-\upsilon}^{\dot{a}+\ddot{a}-\tau}\eth(u)\,du\right| \leq \frac{5(\upsilon-\tau)}{24}(H-h).$$

Corollary 2.15. *Under the condition of Corollary 2.14, suppose there exists a constant $H \in \mathbb{R}^{+}$ such that $\eth'(\Bbbk_1) \leq H$ for all $\Bbbk_1 \in [\dot{a},\ddot{a}]$, then we get*

$$\left|\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\tau)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth(\dot{a}+\ddot{a}-\upsilon)\right]-\frac{1}{\upsilon-\tau}\int_{\dot{a}+\ddot{a}-\upsilon}^{\dot{a}+\ddot{a}-\tau}\eth(u)\,du\right| \leq \frac{5(\upsilon-\tau)}{12}H.$$

Remark 2.16. *Taking $\tau = \dot{a}$ and $\upsilon = \ddot{a}$ in Corollary 2.15, it follows that*

$$\left|\frac{1}{3}\left[2\eth(\dot{a})-\eth\left(\frac{\dot{a}+\ddot{a}}{2}\right)+2\eth(\ddot{a})\right]-\frac{1}{\ddot{a}-\dot{a}}\int_{\dot{a}}^{\ddot{a}}\eth(u)\,du\right| \leq \frac{5(\ddot{a}-\dot{a})}{12}H.$$

This result was derived by Alomari and Liu [28, Theorem 3.1].

Theorem 2.17. *Let $\eth$ be defined as in Theorem 2.3 holds. If $\eth'$ is an L-Lipschitz function on $[\dot{a},\ddot{a}]$, one obtains the following fractional Milne-Mercer type inequality:*

$$\left|\frac{1}{3}\left[2\eth\left(\dot{a}+\ddot{a}-\tau\right)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth\left(\dot{a}+\ddot{a}-\upsilon\right)\right]\right.$$

$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha}}\left[J^{\alpha}_{(\dot{a}+\ddot{a}-\upsilon)^{+}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+J^{\alpha}_{(\dot{a}+\ddot{a}-\tau)^{-}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right]\right|$$

$$\leq\frac{(\upsilon-\tau)^{2}}{4}\left(\frac{1}{6}+\frac{1}{(2+\alpha)}\right)L.$$

Proof. Applying Lemma 2.1 and using the Lipschitz continuity of $\eth'$ with constant L-Lipschitz function, we have

$$\left|\frac{1}{3}\left[2\eth\left(\dot{a}+\ddot{a}-\tau\right)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth\left(\dot{a}+\ddot{a}-\upsilon\right)\right]\right.$$

$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\upsilon-\tau)^{\alpha}}\left[J^{\alpha}_{(\dot{a}+\ddot{a}-\upsilon)^{+}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+J^{\alpha}_{(\dot{a}+\ddot{a}-\tau)^{-}}\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right]\right|$$

$$=\left|\frac{\upsilon-\tau}{4}\int_{0}^{1}\left(\Bbbk_{1}^{\alpha}+\frac{1}{3}\right)\left[\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1+\Bbbk_{1}}{2}\tau+\frac{1-\Bbbk_{1}}{2}\upsilon\right)\right)-\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1-\Bbbk_{1}}{2}\tau+\frac{1+\Bbbk_{1}}{2}\upsilon\right)\right)d\Bbbk_{1}\right]\right|$$

$$\leq\frac{\upsilon-\tau}{4}\int_{0}^{1}\left(\Bbbk_{1}^{\alpha}+\frac{1}{3}\right)\left[\left|\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1+\Bbbk_{1}}{2}\tau+\frac{1-\Bbbk_{1}}{2}\upsilon\right)\right)+\eth'\left(\dot{a}+\ddot{a}-\left(\frac{1-\Bbbk_{1}}{2}\tau+\frac{1+\Bbbk_{1}}{2}\upsilon\right)\right)\right|\right]d\Bbbk_{1}$$

$$\leq\frac{\upsilon-\tau}{4}\int_{0}^{1}\left(\Bbbk_{1}^{\alpha}+\frac{1}{3}\right)L\Bbbk_{1}(\upsilon-\tau)d\Bbbk_{1}$$

$$=\frac{(\upsilon-\tau)^{2}}{4}\left(\frac{1}{6}+\frac{1}{(2+\alpha)}\right)L.$$

Hence, the proof of Theorem 2.17 is concluded. □

Corollary 2.18. *By setting $\alpha=1$ in Theorem 2.17, we obtain*

$$\left|\frac{1}{3}\left[2\eth\left(\dot{a}+\ddot{a}-\tau\right)-\eth\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\eth\left(\dot{a}+\ddot{a}-\upsilon\right)\right]-\frac{1}{\upsilon-\tau}\int_{\dot{a}+\ddot{a}-\upsilon}^{\dot{a}+\ddot{a}-\tau}\eth(u)\,du\right|\leq\frac{(\upsilon-\tau)^{2}}{8}L.$$

3. Applications

3.1. Milne's Quadrature Formula:

Consider a partition of points P of the interval $[\dot{a},\ddot{a}]$ given by $\dot{a}=\mu_0<\mu_1<\cdots<\mu_n=\ddot{a}$, then the corresponding quadrature formula takes the following form:

$$\int_{\dot{a}}^{\ddot{a}}\eth(u)\,du=U(\eth,P)+\Xi(\eth,P),$$

where

$$U(\eth,P)=\sum_{i=0}^{n-1}\frac{(\mu_{i+1}-\mu_i)}{4}\left(2\eth\left(\dot{a}+\ddot{a}-\mu_i\right)-\eth\left(\dot{a}+\ddot{a}-\frac{\mu_i+\mu_{i+1}}{2}\right)+2\eth\left(\dot{a}+\ddot{a}-\mu_{i+1}\right)\right)$$

and $\Upsilon(\eth,P)$ represents the approximation error associated with evaluating the integral $\int_{\dot{a}}^{\ddot{a}}\eth(u)\,du$.

Proposition 3.1. *Under the assumptions of Lemma 2.1 if $|\eth'|$ is a convex function, then the following error bound holds:*

$$|\Upsilon(\eth,P)|\leq\frac{1}{4}\left[\left(\frac{3^{-1-p}\left(4^{p+1}-1\right)}{(p+1)}\right)^{\frac{1}{p}}\right]\sum_{i=0}^{n-1}(\mu_{i+1}-\mu_i)^{2}$$

$$\times\left[\left(\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)-\left(\frac{3\left|\eth'(\mu_{i+1})\right|^q+\left|\eth'(\mu_i)\right|^q}{4}\right)\right)^{\frac{1}{q}}$$

$$+\left(\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)-\left(\frac{3\left|\eth'(\mu_i)\right|^q+\left|\eth'(\mu_{i+1})\right|^q}{4}\right)\right)^{\frac{1}{q}}\Bigg].$$

Proof. Using the Corollary 2.9 on the subintervals $[\mu_i,\mu_{i+1}]\subseteq[\dot{a},\ddot{a}]$, $(i=0,1,2,\ldots,n-1)$ of the partition P, we have

$$\left|\int_{\dot{a}+\ddot{a}-\mu_{i+1}}^{\dot{a}+\ddot{a}-\mu_i}\eth(u)\,du-\frac{1}{3}\left[2\eth(\dot{a}+\ddot{a}-\mu_i)+\eth\left(\dot{a}+\ddot{a}-\frac{\mu_i+\mu_{i+1}}{2}\right)+2\eth(\dot{a}+\ddot{a}-\mu_{i+1})\right]\right|$$

$$\leq\frac{1}{4}\left[\left(\frac{3^{-1-p}\left(4^{p+1}-1\right)}{(p+1)}\right)^{\frac{1}{p}}\right]\sum_{i=0}^{n-1}(\mu_{i+1}-\mu_i)^2$$

$$\times\left[\left(\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)-\left(\frac{3\left|\eth'(\mu_{i+1})\right|^q+\left|\eth'(\mu_i)\right|^q}{4}\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left(\left|\eth'(\dot{a})\right|^q+\left|\eth'(\ddot{a})\right|^q\right)-\left(\frac{3\left|\eth'(\mu_i)\right|^q+\left|\eth'(\mu_{i+1})\right|^q}{4}\right)\right)^{\frac{1}{q}}\right].$$

Summing the above inequality for every $i=0,1,2,\ldots,n-1$, combined with the triangle inequality, we obtain the desired result. □

3.2. q-Digamma Function:

Let us first recall the definition of the q-digamma function along with its mathematical representations. Assume that $0<q<1$. The q-digamma function as introduced in (refer to [29]), is denoted by

$$\Omega_q(z)=-\ln(1-q)+\ln(q)\sum_{i=0}^{\infty}\frac{q^{i+z}}{1-q^{i+z}}$$

$$=-\ln(1-q)+\ln(q)\sum_{i=1}^{\infty}\frac{q^{iz}}{1-q^i}.$$

If $q>1$ and $z>0$, the q-digamma function Ω_q can be expressed as:

$$\Omega_q(z)=-\ln(1-q)+\ln(q)\left[z-\frac{1}{2}-\sum_{i=0}^{\infty}\frac{q^{-(i+z)}}{1-q^{-(i+z)}}\right]$$

$$=-\ln(1-q)+\ln(q)\left[z-\frac{1}{2}-\sum_{i=1}^{\infty}\frac{q^{-iz}}{1-q^{-i}}\right].$$

Proposition 3.2. *Assume that all the conditions of Corollary 2.5 are satisfied, then we have*

$$\left|\frac{1}{3}\left[2\Omega_q'(\dot{a}+\ddot{a}-\tau)-\Omega_q'\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\Omega_q'(\dot{a}+\ddot{a}-\upsilon)\right]-\frac{\Omega_q(\dot{a}+\ddot{a}-\tau)+\Omega_q(\dot{a}+\ddot{a}-\upsilon)}{\upsilon-\tau}\right|$$

$$\leq\frac{\upsilon-\tau}{4}\left[\frac{5}{3}\left(\left|\Omega_q''(\dot{a})\right|+\left|\Omega_q''(\ddot{a})\right|\right)-\frac{5}{6}\left(\left|\Omega_q''(\tau)\right|+\left|\Omega_q''(\upsilon)\right|\right)\right].$$

Proof. If we set $\eth(z)\to\Omega_q'(z)$, the result follows directly. □

Proposition 3.3. *Suppose that all the conditions of Corollary 2.9 are fulfilled, then we get*

$$\left|\frac{1}{3}\left[2\Omega_q'(\dot{a}+\ddot{a}-\tau)-\Omega_q'\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)+2\Omega_q'(\dot{a}+\ddot{a}-\upsilon)\right]-\frac{\Omega_q(\dot{a}+\ddot{a}-\tau)+\Omega_q(\dot{a}+\ddot{a}-\upsilon)}{\upsilon-\tau}\right|$$

$$\leq \frac{\upsilon-\tau}{4}\left[\left(\frac{3^{-1-p}\left(4^{p+1}-1\right)}{(p+1)}\right)^{\frac{1}{p}}\right]\left[\left(\left(\left|\Omega_q''(\dot{a})\right|^q+\left|\Omega_q''(\ddot{a})\right|^q\right)-\left(\frac{3\left|\Omega_q''(\upsilon)\right|^q+\left|\Omega_q''(\tau)\right|^q}{4}\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left|\Omega_q''(\dot{a})\right|^q+\left|\Omega_q''(\ddot{a})\right|^q-\left(\frac{3\left|\Omega_q''(\tau)\right|^q+\left|\Omega_q''(\upsilon)\right|^q}{4}\right)\right)^{\frac{1}{q}}\right].$$

Proof. If we set $\eth(z)\to\Omega_q'(z)$, the result follows immediately. □

3.3. Modified Bessel Function:

Let the first kind of modified Bessel function I_σ, which have the series representation ([30, p.77]):

$$I_\sigma(z)=\sum_{n\geq 0}\frac{\left(\frac{z}{2}\right)^{\sigma+2n}}{n!\Gamma(\sigma+n+1)},$$

where $z\in\mathbb{R}$ and $\sigma>-1$. The second kind of Modified Bessel function $\mathbb{C}_\sigma$ [30, p.77] is defined as:

$$\mathbb{C}_\sigma(z)=\frac{\dot{a}}{2}\frac{I_{-\sigma}(z)-I_\sigma(z)}{\sin\sigma\dot{a}}.$$

Let the function $\Omega_\sigma(z):\mathbb{R}\to[1,\infty)$ defined by

$$\Omega_\sigma(z)=2^\sigma\Gamma(\sigma+1)z^{-\sigma}\mathbb{C}_\sigma(z),$$

The formula for the first-order derivative of $\Omega_\sigma(z)$ is provided in [30, p.77] as:

$$\Omega_\sigma'(z)=\frac{z}{2(\sigma+1)}\Omega_{\sigma+1}(z), \tag{3.1}$$

and second derivative can be easily obtained from equation (3.1) as:

$$\Omega_\sigma''(z)=\frac{z^2}{4(\sigma+1)(\sigma+2)}\Omega_{\sigma+1}(z)+\frac{1}{2(\sigma+1)}\Omega_{\sigma+1}(z).$$

Proposition 3.4. *Let $\dot{a},\ddot{a}\in\mathbb{R}$ and $\sigma>-1$, then we have*

$$\left|\frac{1}{3}\left[\frac{(\dot{a}+\ddot{a}-\tau)}{(\sigma+1)}\Omega_{\sigma+1}(\dot{a}+\ddot{a}-\tau)-\frac{\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)}{2(\sigma+1)}\Omega_{\sigma+1}\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right]\right.$$

$$\left.+\frac{(\dot{a}+\ddot{a}-\upsilon)}{(\sigma+1)}\Omega_{\sigma+1}(\dot{a}+\ddot{a}-\upsilon)-\frac{\Omega_\sigma(\dot{a}+\ddot{a}-\tau)+\Omega_\sigma(\dot{a}+\ddot{a}-\upsilon)}{\upsilon-\tau}\right|$$

$$\leq\frac{\upsilon-\tau}{4}\left[\frac{5}{3}\left(\frac{\dot{a}^2}{4(\sigma+1)(\sigma+2)}\Omega_{\sigma+1}(\dot{a})+\frac{1}{2(\sigma+1)}\Omega_{\sigma+1}(\dot{a})+\frac{\ddot{a}^2}{4(\sigma+1)(\sigma+2)}\Omega_{\sigma+1}(\ddot{a})+\frac{1}{2(\sigma+1)}\Omega_{\sigma+1}(\ddot{a})\right)\right.$$

$$\left.-\frac{5}{6}\left(\frac{\tau^2}{4(\sigma+1)(\sigma+2)}\Omega_{\sigma+1}(\tau)+\frac{1}{2(\sigma+1)}\Omega_{\sigma+1}(\tau)+\frac{\upsilon^2}{4(\sigma+1)(\sigma+2)}\Omega_{\sigma+1}(\upsilon)+\frac{1}{2(\sigma+1)}\Omega_{\sigma+1}(\upsilon)\right)\right].$$

Proof. This result is obtained directly from Corollary 2.5 by setting $\eth(z)=\Omega_\sigma'(z)$ for $z>0$. □

Proposition 3.5. *Let $\dot{a},\ddot{a}\in\mathbb{R}$ and $\sigma>-1$, then we have*

$$\left|\frac{1}{3}\left[\frac{(\dot{a}+\ddot{a}-\tau)}{(\sigma+1)}\Omega_{\sigma+1}(\dot{a}+\ddot{a}-\tau)-\frac{\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)}{2(\sigma+1)}\Omega_{\sigma+1}\left(\dot{a}+\ddot{a}-\frac{\tau+\upsilon}{2}\right)\right]\right.$$

$$\left.+\frac{(\dot{a}+\ddot{a}-\upsilon)}{(\sigma+1)}\Omega_{\sigma+1}(\dot{a}+\ddot{a}-\upsilon)-\frac{\Omega_\sigma(\dot{a}+\ddot{a}-\tau)+\Omega_\sigma(\Pi+\ddot{a}-\upsilon)}{\upsilon-\tau}\right|$$

$$\leq\frac{(\upsilon-\tau)^2}{8}L.$$

Proof. This result is obtained directly from Corollary 2.18 by setting $\eth(z)=\Omega_\sigma'(z)$ for $z>0$. □

4. Simulations

This section provides a collection of graphical representations that demonstrate and support the key findings of the study.

Example 4.1. *Assuming the hypotheses of Corollary 2.5 are satisfied are fulfilled, consider the function* $\eth(u) = \frac{s}{n+2s}u^{\frac{n}{s}+2}$ *defined on* $\mathbb{R}^+$ *for parameters* $n > 1$ *and* $s \geq 1$*, and suppose that* $\eth$ *is convex. In addition, its first derivative,* $\eth'(u) = u^{\frac{n}{s}+1}$ *with* $n > 1$ *and* $s \geq 1$ *is also assumed to be convex. For the particular choices* $\dot{a} = 4, \ddot{a} = 5$ *and* $\tau = 2$*,* $\upsilon = 3$*, we proceed as follows:*

$$\left|\frac{1}{3}\left[\frac{2s}{(n+2s)}(7)^{\frac{n}{s}+2} - \frac{s}{(n+2s)}\left(\frac{13}{2}\right)^{\frac{n}{s}+2} + \frac{2s}{(n+2s)}(6)^{\frac{n}{s}+2}\right] - \frac{s^2}{(n+2s)(n+3s)}\left[7^{\frac{n}{s}+3} - 6^{\frac{n}{s}+3}\right]\right|$$

$$\leq \frac{1}{4}\left[\frac{5}{3}\left((4)^{\frac{n}{s}+1} + (5)^{\frac{n}{s}+1}\right) - \frac{5}{6}\left((2)^{\frac{n}{s}+1} + (2)^{\frac{n}{s}+1}\right)\right]. \tag{4.1}$$

In Example 4.1 the evaluated left-hand term remains strictly smaller than the corresponding right-hand term. In Figure 4.1, displays inequality (4.1) for the parameter ranges $n \in [2,3]$ and $s \in [2,3]$. In Figure 4.2, displays inequality (4.1) for the parameter ranges $n \in [2,3]$. In Figure 4.3, displays inequality (4.1) for the parameter ranges $s \in [2,3]$. In each case, the graphical behavior confirms that the left-hand expression stays below its right-hand counterpart.

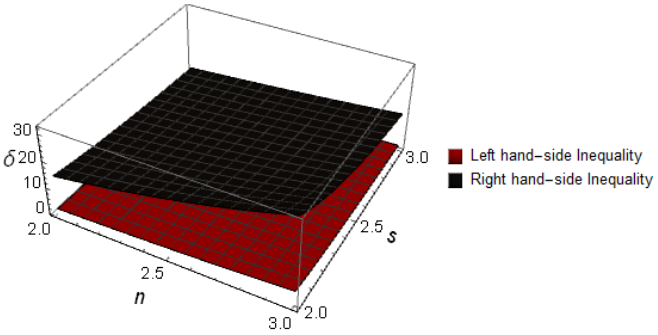

Figure 4.1: We illustrate inequality (4.1) by considering $n \in [2,3]$ and $s \in [2,3]$.

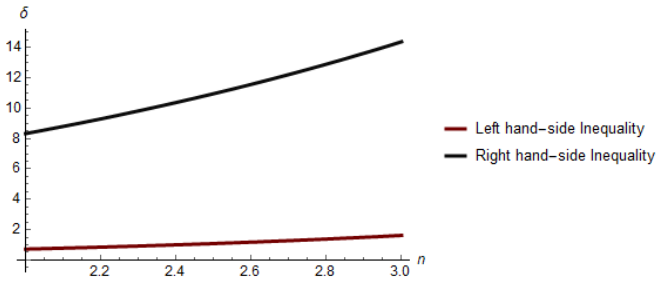

Figure 4.2: We illustrate inequality (4.1) by considering $n \in [2,3]$.

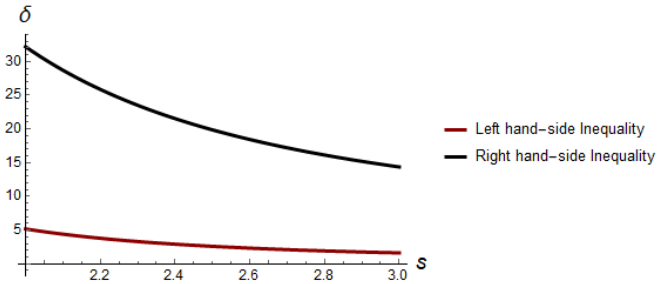

Figure 4.3: We illustrate inequality (4.1) by considering $s \in [2,3]$.

Example 4.2. *Assuming the hypotheses of Corollary 2.9 are satisfied, consider the function* $\eth(u) = \frac{s}{n+2s}u^{\frac{n}{s}+2}$ *defined on* $\mathbb{R}^+$ *for parameters* $n > 1$ *and* $s \geq 1$*, and suppose that* $\eth$ *is convex. In addition, its first derivative,* $\eth'(u) = u^{\frac{n}{s}+1}$ *with* $n > 1$ *and* $s \geq 1$ *is also assumed to be convex. For the particular choices* $\dot{a} = 4, \ddot{a} = 5$ *and* $\tau = 2$*,* $\upsilon = 3$ *with* $p = q = 2$*, we proceed as follows:*

$$\left|\frac{1}{3}\left[\frac{2s}{(n+2s)}(7)^{\frac{n}{s}+2} - \frac{s}{(n+2s)}\left(\frac{13}{2}\right)^{\frac{n}{s}+2} + \frac{2s}{(n+2s)}(6)^{\frac{n}{s}+2}\right] - \frac{s^2}{(n+2s)(n+3s)}\left[7^{\frac{n}{s}+3} - 6^{\frac{n}{s}+3}\right]\right|$$

$$\leq \frac{1}{4}\left(\frac{3^{-3}(4^3-1)}{3}\right)^{\frac{1}{2}}\left[\left(\left((4)^{\frac{n}{s}+1}\right)^2 + \left((5)^{\frac{n}{s}+1}\right)^2 - \left(\frac{3\left((2)^{\frac{n}{s}+1}\right)^2 + \left((3)^{\frac{n}{s}+1}\right)^2}{4}\right)\right)^{\frac{1}{2}}\right.$$

$$\left. + \left(\left((4)^{\frac{n}{s}+1}\right)^2 + \left((5)^{\frac{n}{s}+1}\right)^2 - \left(\frac{\left((2)^{\frac{n}{s}+1}\right)^2 + 3\left((3)^{\frac{n}{s}+1}\right)^2}{4}\right)\right)^{\frac{1}{2}}\right]. \tag{4.2}$$

In Example 4.2 the evaluated left-hand term remains strictly smaller than the corresponding right-hand term. In Figure 4.4, displays inequality (4.2) for the parameter ranges $n \in [2,3]$ and $s \in [2,3]$. In Figure 4.5, displays inequality (4.2) for the parameter ranges $n \in [2,3]$. In Figure 4.6, displays inequality (4.2) for the parameter ranges $s \in [2,3]$. In each case, the graphical behavior confirms that the left-hand expression stays below its right-hand counterpart.

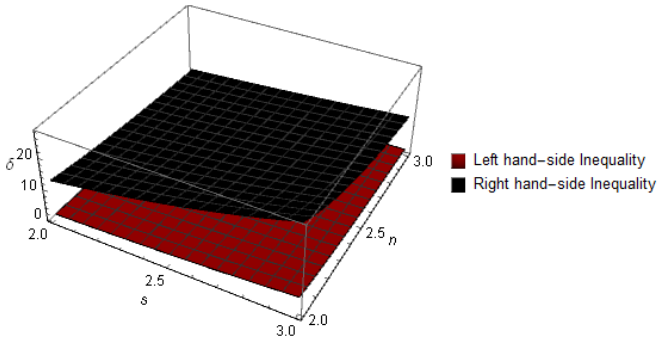

Figure 4.4: We illustrate inequality (4.2) by considering $n \in [2,3]$ and $s \in [2,3]$.

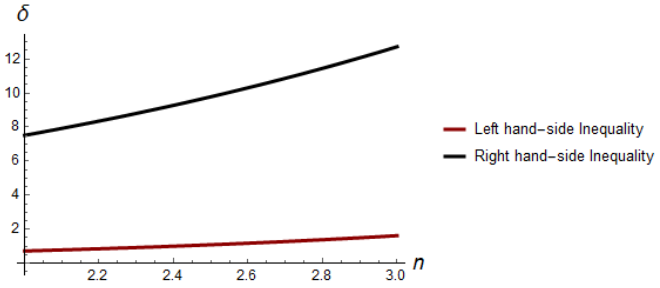

Figure 4.5: We illustrate inequality (4.2) by considering $n \in [2,3]$.

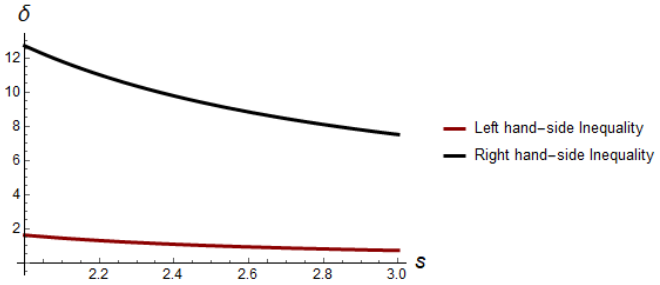

Figure 4.6: We illustrate inequality (4.2) by considering $s \in [2,3]$.

5. Comparative Analysis

In this section, two numerical examples are presented to illustrate the applicability of the obtained Milne–Mercer type inequalities. The functions $\eth(x) = x^2$ and $\eth(x) = x^3$ are considered since they satisfy all the assumptions of the theorem on nonnegative intervals. For several choices of $\dot{a}$, $\ddot{a}$ and τ, υ the exact values of the current result (Corollary 2.5) and previous result (derived by Djenaoui *et al.* [27, Corollary 2.4]) together with their corresponding upper bounds are computed. The accompanying graphs provide a visual comparison, showing that the current result yields a smaller localized error and a tighter upper bound than the corresponding full-interval estimate. Here, E_C and B_C denote the error and error bound for the current result, respectively, while E_P and B_P denote the error and error bound for the previous result, respectively.

Example 5.1. *Let* $\eth(x) = x^2, x \geq 0$. *Then* $\eth'(x) = 2x$, *and* $|\eth'(x)| = 2x$ *is convex on* $[0,\infty)$, *the following numerical verification is obtained.*

No.	$\dot{a}$	$\ddot{a}$	τ	υ	E_C	B_C	E_P	B_P
1	0.0	2.0	0.4	1.4	0.2500	0.9167	1.0000	1.6667
2	1.0	3.0	1.3	2.4	0.3025	1.9708	1.0000	3.3333
3	0.5	2.5	0.8	1.8	0.2500	1.4167	1.0000	2.5000
4	2.0	5.0	2.5	4.0	0.5625	4.6875	2.2500	8.7500
5	1.0	4.0	1.5	3.0	0.5625	3.4375	2.2500	6.2500
6	3.0	6.0	3.4	5.2	0.8100	7.0500	2.2500	11.2500
7	0.2	1.8	0.5	1.3	0.1600	0.7333	0.6400	1.3333
8	4.0	8.0	4.8	7.0	1.2100	11.1833	4.0000	20.0000
9	2.5	4.5	2.9	4.0	0.3025	3.2542	1.0000	5.8333
10	1.5	5.5	2.2	4.6	1.4400	7.2000	4.0000	11.6667

Table 5.1: Comparison of error and error bounds for the function $\eth(x) = x^2$.

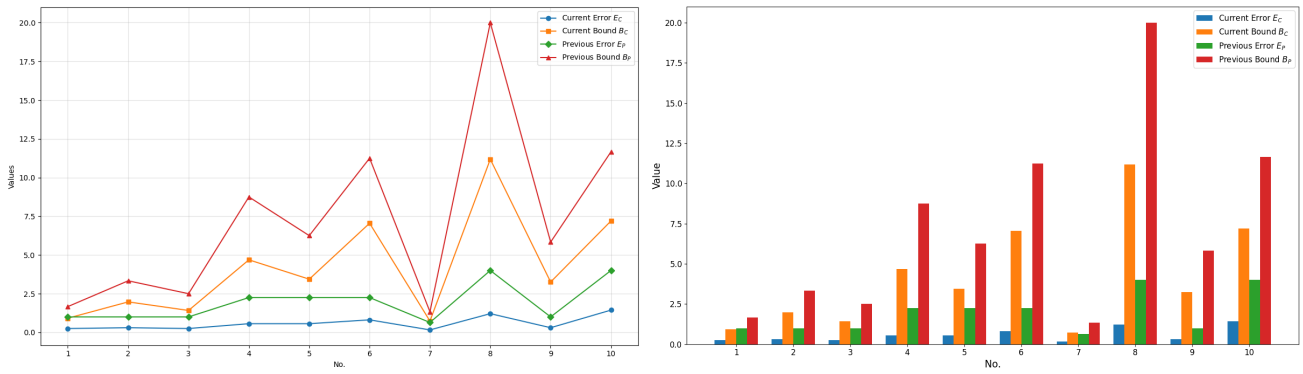

Figure 5.1: Comparison of the actual error and the corresponding upper bounds. The figure illustrates that the current bound is consistently sharper than the previous bound while remaining above the actual error.

Example 5.2. *Let* $\eth(x) = x^3, x \geq 0$. *Then* $\eth'(x) = 3x^2$, *and* $|\eth'(x)| = 3x^2$ *is convex on* $[0,\infty)$, *the following numerical verification is obtained.*

No.	$\dot{a}$	$\ddot{a}$	τ	υ	E_C	B_C	E_P	B_P
1	0.0	2.0	0.4	1.4	0.825000	3.675000	3.000000	5.000000
2	1.0	3.0	1.3	2.4	1.951125	8.628125	6.000000	12.500000
3	0.5	2.5	0.8	1.8	1.275000	5.700000	4.500000	8.125000
4	2.0	5.0	2.5	4.0	6.328125	33.515625	23.625000	54.375000
5	1.0	4.0	1.5	3.0	4.640625	21.328125	16.875000	31.875000
6	3.0	6.0	3.4	5.2	11.421000	57.825000	30.375000	84.375000
7	0.2	1.8	0.5	1.3	0.528000	2.310000	1.920000	3.280000
8	4.0	8.0	4.8	7.0	22.143000	120.945000	72.000000	200.000000
9	2.5	4.5	2.9	4.0	3.221625	19.655625	10.500000	33.125000
10	1.5	5.5	2.2	4.6	15.552000	58.500000	42.000000	81.250000

Table 5.2: Comparison of error and error bounds for the function $\eth(x) = x^3$.

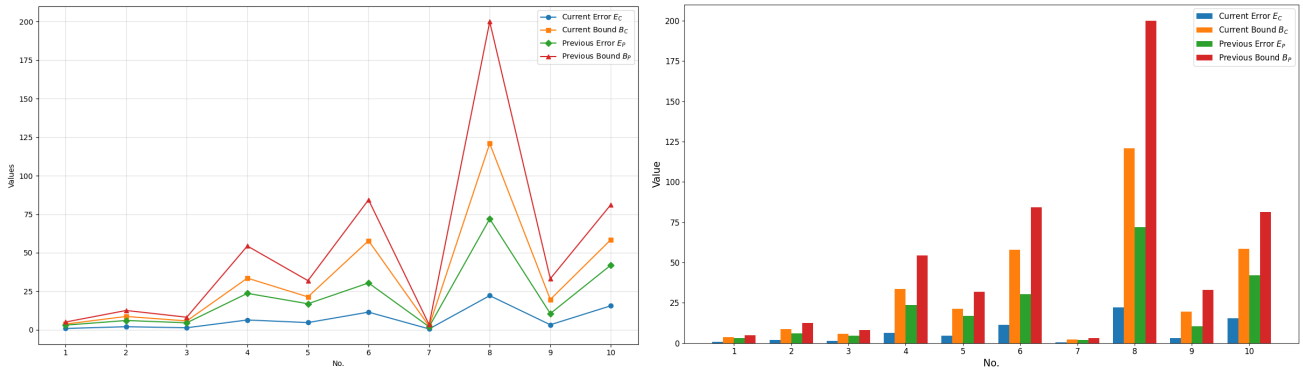

Figure 5.2: Comparison of the actual error and the corresponding upper bounds. The figure illustrates that the current bound is consistently sharper than the previous bound while remaining above the actual error.

6. Conclusions

Recent advancements in the theory of inequalities have primarily focused on developing generalized forms and establishing new bounds for classical inequalities through the use of various fractional integral operators. In the present study, we examined the interplay between Milne-Mercer inequalities and the Riemann-Liouville fractional integral operator for differentiable functions whose derivatives, in absolute value and raised to specific powers, exhibit s-convexity property. Several new special cases of the derived results were discussed, including extensions to bounded and Lipschitz-type functions. To demonstrate the practical effectiveness of the proposed inequalities, graphical illustrations were provided, along with applications to quadrature formulas, q-digamma functions, and modified Bessel functions.

An intriguing direction for future research and interested researchers is to explore whether the techniques developed here can be adapted to further refine Milne-Mercer type inequalities using concepts such as interval analysis, quantum calculus, and coordinate-based approaches. Additionally, future studies may consider extending these results to modified AB-fractional operators and modified Caputo-Fabrizio fractional operators, potentially broadening their applicability in both theoretical and applied contexts.

Article Information

Acknowledgements: The authors wish to thank the editors and reviewers for their valuable comments and suggestions for the betterment of this article.

Author Contributions: Conceptualization: AM, HB; Investigation: AM; Methodology: AM, HB; Formal analysis: AM, AK; Validation: HB, AK, SIB; Visualization: AK, AM; Writing – original draft preparation: AM, SIB; Writing – review & editing: AM, HB, AK, SIB; Supervision: HB, SIB

Artificial Intelligence Statement: No artificial intelligence tools were used in the preparation of this manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the authors.

Plagiarism Statement: This article was scanned by the plagiarism program.

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