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Research Article

On Fractional Newton-Type Method for Nonlinear Problems

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The current manuscript is concerned with the development of the Newton–Raphson method, playing a significant role in mathematics and various other disciplines such as optimization, by using fractional derivatives and fractional Taylor series expansion. The development and modification of the Newton–Raphson method allow us to establish two new methods, which are called first- and second-order fractional Newton–Raphson (FNR) methods. We provide convergence analysis of first- and second-order fractional methods and give a general condition for the convergence of higher-order FNR. Finally, some illustrative examples are considered to confirm the accuracy and effectiveness of both methods.

1. Introduction

Newton–Raphson method is one of the most powerful techniques to locate the solutions of linear or nonlinear equations in numerous areas of science. Compare to the other methods, the convergence rate of this method is much better. In this method, the neighbourhood of the solution of the equations is determined by utilizing the tangent lines of the curves. However, this method has some drawbacks such as having no solution for some forms of equations in a situation that the initial guess or an iteration coincides with a loop which leads to divergence or oscillation of the method.

The theory and applications of the Newton–Raphson method were presented in [1]. Modified versions of Newton's method were given in [2–4]. Newton's method has been implemented for solving constrained or unconstrained minimization problems in [5–7]. A novel block Newton method was developed for the computation invariant pairs to represent eigenvalues and eigenvectors in [8]. A criterion was established for selecting the appropriate model and its applications in [9]. Various optimization methods such as the Newton–Raphson, bisection, gradient, and secant methods were reviewed and discussed in [10]. The optimization solution of the estimating function in the regression models was determined in [11, 12].

In the current study, we focus on developing a novel modification of the Newton–Raphson method by utilizing fractional derivatives and fractional Taylor series expansion which allows us to eliminate the shortcomings of this method. One advantage of this method is that this method can be applied to various fractional derivatives such as Riemann–Liouville and Caputo derivatives [13–22]. Convergence analysis of first- and second-order FNR is provided, and some conditions on initial guess are obtained. Moreover, these conditions are generalized for higher-order FNR. The main advantages of first- and second-order FNR are that they are more effective and accurate compared to other existing methods. Moreover, the convergency of the obtained solutions is faster than the convergency of the solutions, obtained by other methods.

2. Preliminaries

This section is devoted to fundamental notions in fractional calculus [16–18].

Definition 1. β^{th} $(\beta \geq 0)$ order of Riemann–Liouville integral is given by [16]

$$J^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} f(t)\mathrm{d}t, \quad \beta > 0, x > 0,$$
$$J^0 f(x) = f(x). \tag{1}$$

Definition 2. β^{th} order fractional derivative in Caputo sense is given by [16]

$${}^{C}D^{\beta} f(x) = J^{m-\beta} D^m f(x) = \int_0^x (x-t)^{m-\beta-1} \frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t)\mathrm{d}t,$$
$$m-1 < \beta < m, x > 0, \tag{2}$$

where D^m is the ordinary differential operator of order m.

Theorem 1. *[23, 24] Let us suppose that* ${}^{C}D^{j\beta} f(x) \in C([a,b])$ *for* $j = 1, 2, \ldots, n+1$ *where* $\beta \in (0,1]$, *and then, we have*

$$f(x) = \sum_{i=0}^{n} {}^{C}D^{i\beta} f(a) \frac{(x-a)^{i\beta}}{\Gamma(i\beta+1)} + {}^{C}D^{(n+1)\beta} f(\xi) \frac{(x-a)^{(n+1)\beta}}{\Gamma((n+1)\beta+1)}, \tag{3}$$

with $a \le \xi \le x$, *for all* $x \in (a,b]$ *where* ${}^{C}D^{n\beta} = {}^{C}D^{\beta} \cdot {}^{C}D^{\beta}, \ldots, {}^{C}D^{\beta}$ (n *times*). *Notice that the property of Riemann–Liouville derivative of constant is different than zero unlike the Caputo derivative.*

3. Fractional Newton–Raphson (FNR) Method

In this section, we give the formulation of the first- and second-order FNR [25–28].

3.1. First-Order FNR. By taking the first two terms of fractional Taylor series expansion, we have

$$y - f(x_0) = \frac{f^{(\beta)}(x_0)}{\Gamma(\beta+1)} (x - x_0)^{\beta}. \tag{4}$$

To solve for the x intercept, we set $y = 0$ and rearrange the terms

$$0 = f(x_0) + \frac{f^{(\beta)}(x_0)}{\Gamma(\beta+1)} (x_1 - x_0)^{\beta}. \tag{5}$$

Thus, we have

$$x_1 = x_0 + \left(-\frac{f(x_0)}{f^{(\beta)}(x_0)} \Gamma(\beta+1) \right)^{(1/\beta)}. \tag{6}$$

Repetition of this algorithm generates a sequence of x values $x_0, x_1, x_2, \ldots$ which leads to the following recurrence relation:

$$x_{n+1} = x_n + \left(-\frac{f(x_n)}{f^{(\beta)}(x_n)} \Gamma(\beta+1) \right)^{(1/\beta)}, \quad f^{(\beta)}(x_n) \neq 0. \tag{7}$$

3.2. Second-Order FNR. By taking the first three terms of the fractional Taylor series expansion, we get

$$y - f(x_0) = \frac{f^{(\beta)}(x_0)}{\Gamma(\beta+1)} (x - x_0)^{\beta} + \frac{f^{(2\beta)}(x_0)}{\Gamma(2\beta+1)} (x - x_0)^{2\beta}. \tag{8}$$

To solve for the x intercept, we set $y = 0$ and rearrange terms

$$0 = f(x_0) + \frac{f^{(\beta)}(x_0)}{\Gamma(\beta+1)} (x_1 - x_0)^{\beta} + \frac{f^{(2\beta)}(x_0)}{\Gamma(2\beta+1)} (x_1 - x_0)^{2\beta}, \tag{9}$$

which leads to the following

$$x_1 = x_0 + \left(\frac{\Gamma(2\beta+1) \left[-\left(f^{(\beta)}(x_0)/\Gamma(\beta+1) \right) + \sqrt{\left(f^{(\beta)}(x_0)/\Gamma(\beta+1) \right)^2 - 4\left(f^{(2\beta)}(x_0)/\Gamma(2\beta+1) \right) f(x_0)} \right]}{2 f^{(2\beta)}(x_0)} \right)^{1/\beta}. \tag{10}$$

Repetition of this algorithm generates a sequence of x values $x_0, x_1, x_2, \ldots$ which are formulated by the following recurrence relation:

$$x_{n+1} = x_n + \left(\frac{\Gamma(2\beta+1) \left[-\left(f^{(\beta)}(x_n)/\Gamma(\beta+1) \right) + \sqrt{\left(\left(f^{(\beta)}(x_n)/\Gamma(\beta+1) \right) \right)^2 - 4\left(f^{(2\beta)}(x_n)/\Gamma(2\beta+1) \right) f(x_n)} \right]}{2 f^{(2\beta)}(x_n)} \right)^{1/\beta}. \tag{11}$$

4. Convergence

In this section, the convergence analysis of first- and second-order FNR is given by the following theorems, respectively.

Theorem 2. *Assume that* f *is twice continuously differentiable on an open interval* (a,b) *and that there exists a point* $x^* \in (a,b)$ *with* $f^{(\beta)}(x^*) \neq 0$. *Implementing the first-order FNR method, we have the following recurrence relation:*

$$x_{n+1} = x_n + \left(-\frac{f(x_n)}{f^{(\beta)}(x_n)} \Gamma(\beta+1) \right)^{1/\beta}, \quad n = 1, 2, \ldots. \tag{12}$$

Under the assumption that x_n converges to x^* as $n \longrightarrow \infty$, we have

$$\begin{aligned} \left| x_{n+1} - x^* \right| &\leq M \left| x_n - x^* \right|^2, \\ M &> \left(\frac{\left| f^{(2\beta)}(x^*) \right|}{\left| f^{(\beta)}(x^*) \right|} \frac{\Gamma(\beta+1)}{\Gamma(2\beta+1)} \right)^{1/\beta}, \end{aligned} \tag{13}$$

for n sufficiently large. Thus, x_n converges to x^* quadratically.

Proof. Let $e_n = x_n - x^*$, so that $x_n - e_n = x^*$. Setting $x = x_n$ and $h = -e_n$ in fractional Taylor's Theorem, we get

$$f(x_n - e_n) = f(x_n) - \frac{e_n^{\beta}}{\Gamma(\beta+1)} f^{(\beta)}(x_n) + \frac{e_n^{2\beta}}{\Gamma(2\beta+1)} f^{(2\beta)}(\xi_n), \tag{14}$$

for some $\xi_n \in (x_n, x^*)$. Since $f(x_n - e_n) = f(x^*) = 0$, we have

$$0 = f(x_n) - \frac{(x_n - x^*)^{\beta}}{\Gamma(\beta+1)} f^{(\beta)}(x_n) + \frac{e_n^{2\beta}}{\Gamma(2\beta+1)} f^{(2\beta)}(\xi_n). \tag{15}$$

Having the condition that β^{th} derivative of f is continuous with $f^{(\beta)}(x_n) \neq 0$ as long as x_n is close enough to x^* allows us to divide by $(f^{\beta}(x_n)/\Gamma(\beta+1))$ which leads to the following:

$$0 = \frac{f(x_n)}{f^{(\beta)}(x_n)} - (x_n - x^*)^{\beta} + \frac{e_n^{2\beta}}{f^{(2\beta)}(\xi_n)} \frac{\Gamma(\beta+1)}{\Gamma(2\beta+1)}. \tag{16}$$

As a result, the formulation of first-order FNR gives the following:

$$(x_{n+1} - x^*)^{\beta} = e_n^{2\beta} \frac{f^{(2\beta)}(\xi_n)}{f^{(\beta)}(x_n)} \frac{\Gamma(\beta+1)}{\Gamma(2\beta+1)}. \tag{17}$$

After rearrangement, we have

$$\left| x_{n+1} - x^* \right| = \left| e_n^2 \right| \left(\frac{\left| f^{(2\beta)}(\xi_n) \right|}{\left| f^{(\beta)}(x_n) \right|} \frac{\Gamma(\beta+1)}{\Gamma(2\beta+1)} \right)^{1/\beta}. \tag{18}$$

Finally,

$$\left| x_{n+1} - x^* \right| = \left(\frac{\left| f^{(2\beta)}(\xi_n) \right|}{\left| f^{(\beta)}(x_n) \right|} \frac{\Gamma(\beta+1)}{\Gamma(2\beta+1)} \right)^{1/\beta} \left| x_n - x^* \right|^2. \tag{19}$$

By continuity, $f^{(\beta)}(x_n)$ converges to $f^{\beta}(x^*)$. Convergence of $\xi_n \in (x_n, x^*)$ to x^* leads to convergence of $f^{(2\beta)}(\xi_n)$ to $f^{(2\beta)}(x^*)$. As a result, we have

$$\left| x_{n+1} - x^* \right| \leq M \left| x_n - x^* \right|^2, \tag{20}$$

if

$$\begin{aligned} M &> \left(\frac{\left| f^{(2\beta)}(x^*) \right|}{\left| f^{(\beta)}(x^*) \right|} \frac{\Gamma(\beta+1)}{\Gamma(2\beta+1)} \right)^{1/\beta} \\ \text{or } M^{\beta} \frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)} &> \frac{\left| f^{(2\beta)}(x^*) \right|}{\left| f^{(\beta)}(x^*) \right|}, \end{aligned} \tag{21}$$

for sufficiently large n. □

Theorem 3. *Assume that f is three times continuously differentiable on an open interval (a, b) and that there exists $x^* \in (a, b)$ with $f^{(2\beta)}(x^*) \neq 0$. Implementing the first-order FNR method, we have the following recurrence relation:*

$$x_{n+1} = x_n + \left(\frac{\Gamma(2\beta+1) \left[-\left(f^{(\beta)}(x_n)/\Gamma(\beta+1) \right) + \sqrt{\left(f^{(\beta)}(x_n)/\Gamma(\beta+1) \right)^2 - 4 \left(f^{(2\beta)}(x_n)/\Gamma(2\beta+1) \right) f(x_n)} \right]}{2 f^{(2\beta)}(x_n)} \right)^{1/\beta}, \quad n = 1, 2, \ldots. \tag{22}$$

Under the assumption that x_n converges to x^* as $n \longrightarrow \infty$, we have

$$\begin{aligned} \left| x_{n+1} - x^* \right| &\leq M \left| x_n - x^* \right|^{3/2}, \\ M &> \left(\frac{\left| f^{(3\beta)}(x^*) \right|}{\left| f^{(2\beta)}(x^*) \right|} \frac{\Gamma(2\beta+1)}{\Gamma(3\beta+1)} \right)^{1/2\beta}, \end{aligned} \tag{23}$$

for sufficiently large n. Thus, x_n converges to x^* quadratically.

Proof. Let $e_n = x_n - x^*$, so that $x_n - e_n = x^*$. Setting $x = x_n$ and $h = -e_n$, in fractional Taylor's Theorem, we get

$$f(x_n - e_n) = f(x_n) - \frac{e_n^{\beta}}{\Gamma(\beta+1)} f^{(\beta)}(x_n) + \frac{e_n^{2\beta}}{\Gamma(2\beta+1)} f^{(2\beta)}(x_n) - \frac{e_n^{3\beta}}{\Gamma(3\beta+1)} f^{(2\beta)}(\xi_n), \tag{24}$$

Table 1: First-order FNR and second-order FNR results for $f_1(x)$ with Caputo derivative and initial guess $x_0 = 1$.

	β	x_{n+1}	$\lvert x_{n+1} - x_n\rvert$	$\lVert f(x_{n+1})\rVert$	n
1st N-R	1	4.00000000	1.46549$e-14$	2.84217$e-14$	10
2nd N-R	1	4.00000000	8.38330$e-18$	0.00000$e+00$	8
1st N-R	0.98	4.00000000	9.65937$e-09$	1.23761$e-08$	17
2nd N-R	0.98	4.00000000	8.93729$e-09$	1.14804$e-08$	17
1st N-R	0.96	3.99999999	6.31251$e-09$	2.16859$e-08$	28
2nd N-R	0.96	3.99999999	8.36562$e-09$	2.82381$e-08$	27
1st N-R	0.94	3.99999997	7.76010$e-09$	5.26373$e-08$	41
2nd N-R	0.94	3.99999997	8.85140$e-09$	5.94304$e-08$	40
1st N-R	0.92	3.99999994	9.40705$e-09$	1.13137$e-07$	58
2nd N-R	0.92	3.99999995	8.97743$e-09$	1.08442$e-07$	57
1st N-R	0.9	3.99999990	9.68422$e-09$	1.98513$e-07$	82
2nd N-R	0.9	3.99999990	9.48429e-09	1.94823$e-07$	80
1st N-R	0.88	3.99999984	9.37726e-09	3.20376$e-07$	116
2nd N-R	0.88	3.99999983	9.85311$e-09$	3.34523$e-07$	112
1st N-R	0.86	3.99999973	9.78559$e-09$	5.41971$e-07$	162
2nd N-R	0.86	3.99999973	9.66965$e-09$	5.36477$e-07$	158
1st N-R	0.84	3.99999956	9.88726$e-09$	8.80428$e-07$	227
2nd N-R	0.84	3.99999956	9.82503e-09	8.75790$e-07$	222
1st N-R	0.82	3.99999930	9.84707$e-09$	1.40112$e-06$	319
2nd N-R	0.82	3.99999930	9.85142$e-09$	1.40163$e-06$	313
1st N-R	0.8	3.99999888	9.90469e-09	2.23380$e-06$	448
2nd N-R	0.8	3.99999888	9.93908$e-09$	2.23999$e-06$	441
1st N-R	0.78	3.99999600	2.83959$e-08$	8.00875$e-06$	500
2nd N-R	0.78	3.99999620	2.65292$e-08$	7.59583$e-06$	500
1st N-R	0.76	3.99998275	1.10103$e-07$	3.44952$e-05$	500
2nd N-R	0.76	3.99998367	1.02423$e-07$	3.26542$e-05$	500
1st N-R	0.74	3.99993896	3.53606$e-07$	1.22080$e-04$	500
2nd N-R	0.74	3.99994240	3.26873$e-07$	1.15193$e-04$	500
1st N-R	0.72	3.99981521	9.78747$e-07$	3.69503$e-04$	500
2nd N-R	0.72	3.99982645	8.96989$e-07$	3.47038$e-04$	500
1st N-R	0.7	3.99950671	2.40749$e-06$	9.86099$e-04$	500
2nd N-R	0.7	3.99953965	2.18121$e-06$	9.20282$e-04$	500

for some $\xi_n \in (x_n, x^*)$. Since $f(x_n - e_n) = f(x^*) = 0$, we have

$$0 = f(x_n) - \frac{(x_n - x^*)^{\beta}}{\Gamma(\beta+1)} f^{(\beta)}(x_n) + \frac{(x_n - x^*)^{2\beta}}{\Gamma(2\beta+1)} f^{(2\beta)}(x_n) - \frac{(x_n - x^*)^{3\beta}}{\Gamma(3\beta+1)} f^{(3\beta)}(\xi_n). \tag{25}$$

Having the condition that β^{th} derivative of f is continuous with as long as x_n is close enough to x^* allows us to divide by $f^{2\beta}(x_n)/\Gamma(2\beta+1)$ which leads to the following:

$$0 = \frac{\Gamma(2\beta+1) f(x_n)}{f^{(2\beta)}(x_n)} - \frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)} \frac{(x_n - x^*)^{\beta} f^{(\beta)}(x_n)}{f^{(2\beta)}(x_n)} + (x_n - x^*)^{2\beta} - e_n^{3\beta} \frac{f^{(3\beta)}(\xi_n)}{f^{(2\beta)}(x_n)} \frac{\Gamma(2\beta+1)}{\Gamma(3\beta+1)}. \tag{26}$$

As a result, the formulation of first-order FNR gives the following:

$$(x_{n+1} - x^*)^{2\beta} = e_n^{3\beta} \frac{f^{(3\beta)}(\xi_n)}{f^{(2\beta)}(x_n)} \frac{\Gamma(2\beta+1)}{\Gamma(3\beta+1)}. \tag{27}$$

After rearrangement, we have

$$\lvert x_{n+1} - x^*\rvert = \lvert e_n^{3/2}\rvert \left(\frac{\lvert f^{(3\beta)}(\xi_n)\rvert}{\lvert f^{(2\beta)}(x_n)\rvert} \frac{\Gamma(2\beta+1)}{\Gamma(3\beta+1)} \right)^{(1/2\beta)}. \tag{28}$$

Finally,

$$\lvert x_{n+1} - x^*\rvert = \left(\frac{\lvert f^{(3\beta)}(\xi_n)\rvert}{\lvert f^{(2\beta)}(x_n)\rvert} \frac{\Gamma(2\beta+1)}{\Gamma(3\beta+1)} \right)^{1/2\beta} \lvert x_n - x^*\rvert^{3/2}. \tag{29}$$

By continuity, $f^{(2\beta)}(x_n)$ converges to $f^{2\beta}(x^*)$. Convergence of $\xi_n \in (x_n, x^*)$ to x^* leads to convergence of $f^{(3\beta)}(\xi_n)$ to $f^{(3\beta)}(x^*)$. As a result, we have

$$\lvert x_{n+1} - x^*\rvert \le M \lvert x_n - x^*\rvert^{3/2}, \tag{30}$$

if

$$M > \left(\frac{\lvert f^{(3\beta)}(x^*)\rvert}{\lvert f^{(2\beta)}(x^*)\rvert} \frac{\Gamma(2\beta+1)}{\Gamma(3\beta+1)} \right)^{1/2\beta} \quad \text{or } M^{2\beta} \frac{\Gamma(3\beta+1)}{\Gamma(2\beta+1)} > \frac{\lvert f^{(3\beta)}(x^*)\rvert}{\lvert f^{(2\beta)}(x^*)\rvert}, \tag{31}$$

for sufficiently large n.

In general, for the convergence of higher-order FNR, we obtain the following condition:

Table 2: First-order FNR and second-order FNR results for $f_1(x)$ with Riemann–Liouville derivative and initial estimation $x_0 = 1$.

	β	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$	$\lVert f(x_{n+1}) \rVert$	n
1st N-R	1	4.00000000	1.46549e − 14	2.84217e − 14	10
2nd N-R	1	4.00000000	8.38330e − 18	0.00000e + 00	8
1st N-R	0.98	4.00000000	4.68492e − 09	4.77773e − 09	16
2nd N-R	0.98	4.00000000	6.74956e − 09	6.72255e − 09	15
1st N-R	0.96	3.99999999	9.31417e − 09	2.26436e − 08	20
2nd N-R	0.96	3.99999999	6.15820e − 09	1.54268e − 08	21
1st N-R	0.94	3.99999999	8.41656e − 09	3.90606e − 08	28
2nd N-R	0.94	3.99999998	9.57249e − 09	4.39009e − 08	28
1st N-R	0.92	3.99999997	9.83530e − 09	7.65395e − 08	35
2nd N-R	0.92	3.99999997	8.10233e − 09	6.42359e − 08	38
1st N-R	0.9	3.99999994	9.40993e − 09	1.18506e − 07	49
2nd N-R	0.9	3.99999994	9.37709e − 09	1.18142e − 07	47
1st N-R	0.88	3.99999991	9.61954e − 09	1.88983e − 07	65
2nd N-R	0.88	3.99999990	9.84849e − 09	1.92880e − 07	64
1st N-R	0.86	3.99999986	9.60193e − 09	2.89676e − 07	86
2nd N-R	0.86	3.99999985	9.96806e − 09	2.99047e − 07	85
1st N-R	0.84	3.99999978	9.53412e − 09	4.36870e − 07	114
2nd N-R	0.84	3.99999978	9.64049e − 09	4.40927e − 07	113
1st N-R	0.82	3.99999966	9.94288e − 09	6.80455e − 07	150
2nd N-R	0.82	3.99999966	9.91663e − 09	6.78992e − 07	149
1st N-R	0.8	3.99999949	9.98970e − 09	1.02156e − 06	199
2nd N-R	0.8	3.99999950	9.78692e − 09	1.00502e − 06	199
1st N-R	0.78	3.99999924	9.96299e − 09	1.51847e − 06	265
2nd N-R	0.78	3.99999924	9.98659e − 09	1.52126e − 06	264
1st N-R	0.76	3.99999888	9.90426e − 09	2.24483e − 06	354
2nd N-R	0.76	3.99999887	9.98421e − 09	2.25855e − 06	353
1st N-R	0.74	3.99999833	9.92379e − 09	3.33049e − 06	473
2nd N-R	0.74	3.99999833	9.94614e − 09	3.33603e − 06	473
1st N-R	0.72	3.99999551	2.29296e − 08	8.98833e − 06	500
2nd N-R	0.72	3.99999548	2.30961e − 08	9.03519e − 06	500
1st N-R	0.7	3.99998762	5.73722e − 08	2.47620e − 05	500
2nd N-R	0.7	3.99998750	5.81547e − 08	2.49976e − 05	500

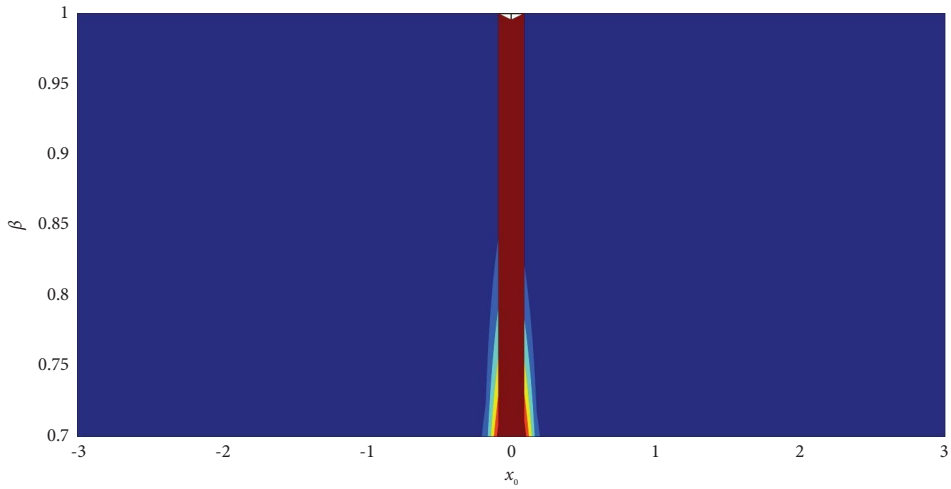


Figure 1: Convergence planes of first-order FNR on $f_1(x)$ for Caputo derivative.

$$\left|x_{n+1} - x^*\right| \le M\left|x_n - x^*\right|^{(k+1/k)},$$

$$M > \left(\left|\frac{f^{((k+1)\beta)}(x^*)}{f^{(k\beta)}(x^*)}\right| \frac{\Gamma(k\beta+1)}{\Gamma((k+1)\beta+1)}\right)^{1/k\beta}, \quad k = 1, 2, 3, \ldots, \tag{32}$$

for sufficiently large n. □

5. Numerical Examples

In this section, some illustrative examples are presented to show the implementation of first- and second-order FNR which allows us to confirm the obtained results given in the previous section. Matlab R2016b with stopping criterion $|x_{n+1} - x_n| < 10^{-8}$ and a maximum of 500 iterations are

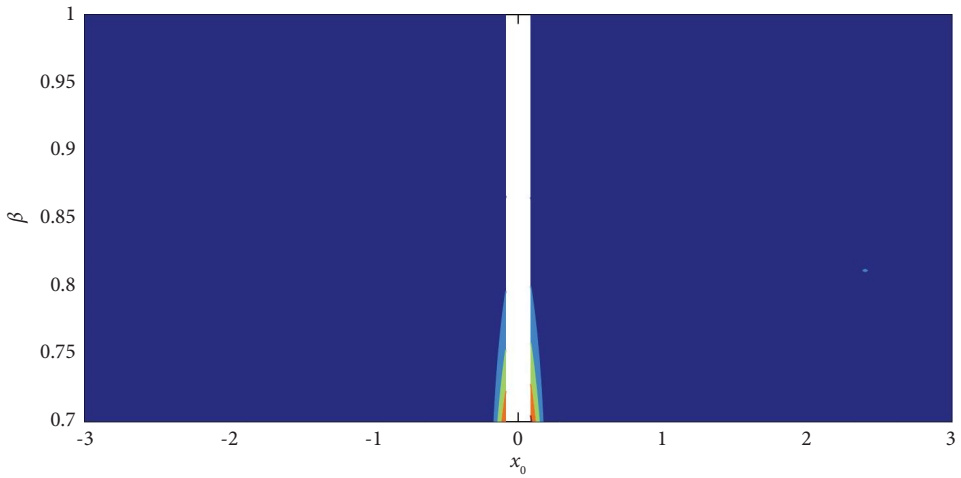


Figure 2: Convergence planes of second-order FNR on $f_1(x)$ for Caputo derivative.

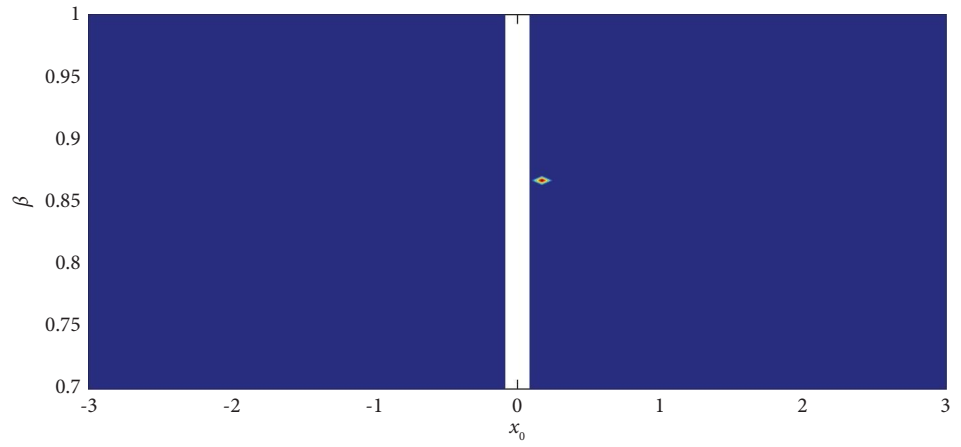


Figure 3: Convergence planes of first-order FNR on $f_1(x)$ for Riemann–Liouville derivative.

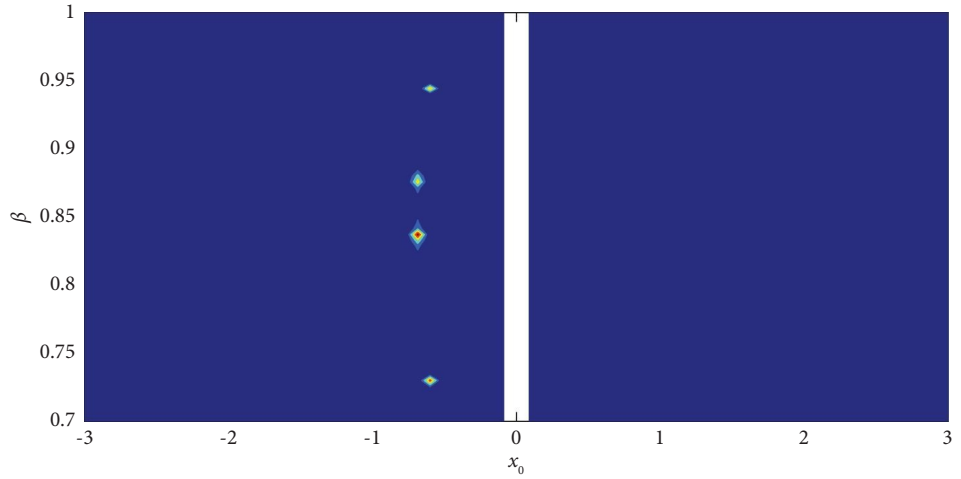


Figure 4: Convergence planes of second-order FNR on $f_1(x)$ for Riemann–Liouville derivative.

utilized. In the tables of corresponding examples, the reached root $|x_{n+1} - x_n|$, $\|f(x_{n+1})\|$, and the number of iterations are shown.

Example 1. Let us consider the function $f_1(x) = x^3 - 10x^2 + 34x - 40$ with roots $x_0 = 4, x_1 = 3 - i$ and $x_2 = 3 + i$.

Table 3: First-order NR and second-order NR results for $f_2(x)$ with Caputo derivative and initial estimation $x_0 = 0.2$.

	β	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$	$\lVert f(x_{n+1}) \rVert$	n
1st N-R	1	−0.00000000	0.00000e + 00	0.00000e + 00	5
2nd N-R	1	−0.00000000	6.66798e − 10	0.00000e + 00	3
1st N-R	0.98	−0.00000000	2.42649e − 09	1.64532e − 10	8
2nd N-R	0.98	−0.00000000	1.07859e − 09	7.31359e − 11	8
1st N-R	0.96	−0.00000000	3.91885e − 09	5.46557e − 10	10
2nd N-R	0.96	−0.00000000	2.58893e − 09	3.61075e − 10	10
1st N-R	0.94	−0.00000000	5.70944e − 09	1.22846e − 09	12
2nd N-R	0.94	−0.00000000	4.32641e − 09	9.30883e − 10	12
1st N-R	0.92	−0.00000000	2.76674e − 09	8.16363e − 10	15
2nd N-R	0.92	−0.00000000	8.17968e − 09	2.41352e − 09	14
1st N-R	0.9	−0.00000000	7.94065e − 09	3.01220e − 09	17
2nd N-R	0.9	−0.00000000	6.59632e − 09	2.50224e − 09	17
1st N-R	0.88	−0.00000000	4.85622e − 09	2.27344e − 09	21
2nd N-R	0.88	−0.00000000	9.82148e − 09	4.59793e − 09	20
1st N-R	0.86	−0.00000000	7.02051e − 09	3.94299e − 09	25
2nd N-R	0.86	−0.00000000	5.90703e − 09	3.31762e − 09	25
1st N-R	0.84	−0.00000000	6.69774e − 09	4.42000e − 09	31
2nd N-R	0.84	−0.00000000	9.87756e − 09	6.51845e − 09	30
1st N-R	0.82	−0.00000001	8.16846e − 09	6.23332e − 09	39
2nd N-R	0.82	−0.00000001	6.60392e − 09	5.03942e − 09	39
1st N-R	0.8	−0.00000001	7.97894e − 09	6.95120e − 09	52
2nd N-R	0.8	−0.00000001	8.52364e − 09	7.42575e − 09	51
1st N-R	0.78	−0.00000001	9.79061e − 09	9.63595e − 09	74
2nd N-R	0.78	−0.00000001	8.49008e − 09	8.35597e − 09	73
1st N-R	0.76	−0.00000001	9.85552e − 09	1.08611e − 08	125
2nd N-R	0.76	−0.00000001	9.15524e − 09	1.00893e − 08	121
1st N-R	0.74	−0.00000001	9.63986e − 09	1.18037e − 08	360
2nd N-R	0.74	−0.00000001	9.78282e − 09	1.19788e − 08	317
1st N-R	0.72	−4.31365378	5.71734e − 07	5.23966e − 05	500
2nd N-R	0.72	−0.14393292	1.11043e − 01	1.43663e − 01	500
1st N-R	0.7	−4.78460364	2.45965e − 01	1.47957e + 00	500
2nd N-R	0.7	−1.29422312	7.32260e − 01	7.50525e − 01	500

Table 4: First-order NR and second-order NR results for $f_2(x)$ with Riemann–Liouville derivative and initial estimation $x_0 = 0.2$.

	β	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$	$\lVert f(x_{n+1}) \rVert$	n
1st N-R	1	0.00000000	0.00000e + 00	0.00000e + 00	5
2nd N-R	1	0.00000000	6.66798e − 10	0.00000e + 00	3
1st N-R	0.98	−0.00000000	2.42649e − 09	1.64532e − 10	8
2nd N-R	0.98	−0.00000000	9.29969e − 10	5.95854e − 11	8
1st N-R	0.96	−0.00000000	3.91885e − 09	5.46557e − 10	10
2nd N-R	0.96	−0.00000000	2.40538e − 09	3.14007e − 10	10
1st N-R	0.94	−0.00000000	5.70944e − 09	1.22846e − 09	12
2nd N-R	0.94	−0.00000000	4.82173e − 09	9.60702e − 10	12
1st N-R	0.92	−0.00000000	2.76674e − 09	8.16363e − 10	15
2nd N-R	0.92	−0.00000000	3.43115e − 09	9.26171e − 10	15
1st N-R	0.9	−0.00000000	7.94065e − 09	3.01220e − 09	17
2nd N-R	0.9	−0.00000000	5.89937e − 09	2.01913e − 09	18
1st N-R	0.88	−0.00000000	4.85622e − 09	2.27344e − 09	21
2nd N-R	0.88	−0.00000000	4.71244e − 09	1.95934e − 09	23
1st N-R	0.86	−0.00000000	7.02051e − 09	3.94299e − 09	25
2nd N-R	0.86	−0.00000000	5.93748e − 09	2.90861e − 09	30
1st N-R	0.84	−0.00000000	6.69774e − 09	4.42000e − 09	31
2nd N-R	0.84	−0.00000000	7.64084e − 09	4.30750e − 09	43
1st N-R	0.82	−0.00000001	8.16846e − 09	6.23332e − 09	39
2nd N-R	0.82	−0.00000001	8.48988e − 09	5.40319e − 09	79
1st N-R	0.8	−0.00000001	7.97894e − 09	6.95120e − 09	52
2nd N-R	0.8	−0.00506018	1.02936e − 02	7.19943e − 03	500
1st N-R	0.78	−0.00000001	9.79061e − 09	9.63595e − 09	74

Table 4: Continued.

	β	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$	$\lVert f(x_{n+1}) \rVert$	n
2nd N-R	0.78	−0.07024016	1.47568$e-01$	9.83535$e-02$	500
1st N-R	0.76	−0.00000001	9.85552$e-09$	1.08611$e-08$	125
2nd N-R	0.76	−0.14147226	2.83238$e-01$	1.86505$e-01$	500
1st N-R	0.74	−0.00000001	9.63986$e-09$	1.18037$e-08$	360
2nd N-R	0.74	−1.57171790	2.40309$e+00$	1.13046$e+00$	500
1st N-R	0.72	−4.31365378	5.71734$e-07$	5.23966$e-05$	500
2nd N-R	0.72	−0.00000000	9.36668$e-09$	5.10297$e-09$	157
1st N-R	0.7	−4.78460364	2.45965$e-01$	1.47957$e+00$	500
2nd N-R	0.7	0.00000000	9.82670$e-09$	5.30280$e-09$	101
1st N-R	0.68	−5.71070798	5.01626$e-01$	1.03562$e+01$	500
2nd N-R	0.68	0.00000000	8.29674$e-09$	4.54705$e-09$	85
1st N-R	0.66	−754.27892554	7.80349$e+01$	2.58569$e+18$	500
2nd N-R	0.66	−0.00000000	8.34916$e-09$	4.76503e09	82
1st N-R	0.64	−1411860.53657482	1.42625$e+05$	3.96270$e+44$	500
2nd N-R	0.64	0.00000000	9.54192$e-09$	5.80404$e-09$	89
1st N-R	0.62	−1063648380.57639720	1.04707$e+08$	4.10943$e+67$	500
2nd N-R	0.62	−0.00000000	8.55254$e-09$	5.65282$e-09$	113
1st N-R	0.6	−875382037053.40283000	8.39601$e+10$	8.59364$e+90$	500
2nd N-R	0.6	−3.22839026	2.60655$e-07$	1.73936$e-04$	500

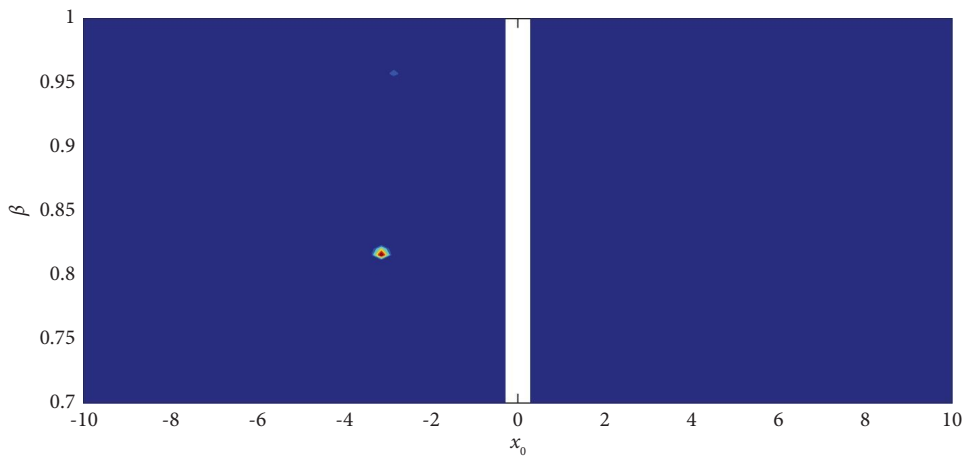


Figure 5: Convergence planes of first-order NR on $f_2(x)$ for Caputo derivative.

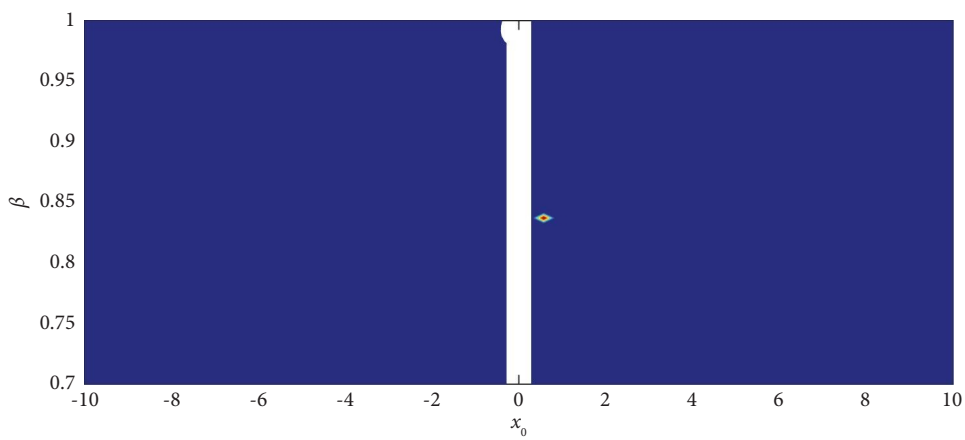


Figure 6: Convergence planes of second-order NR on $f_2(x)$ for Caputo derivative.

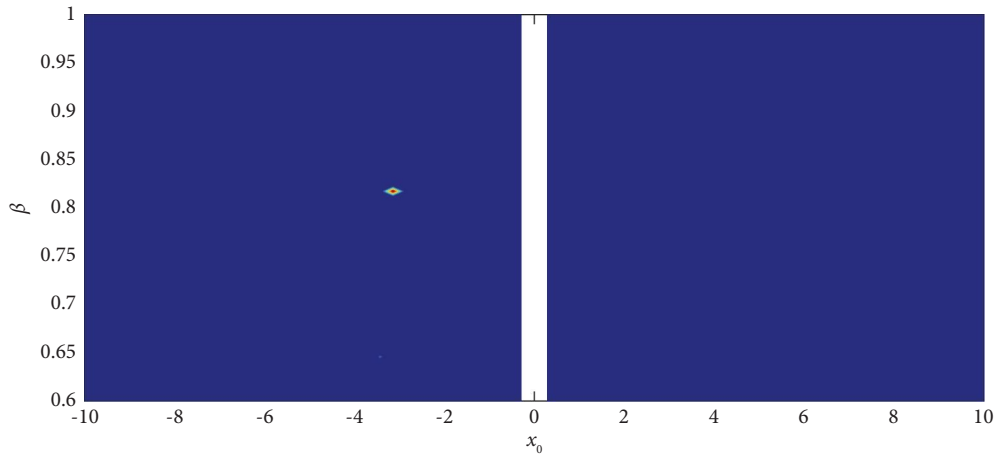


Figure 7: Convergence planes of first-order NR on $f_2(x)$ for Riemann–Liouville derivative.

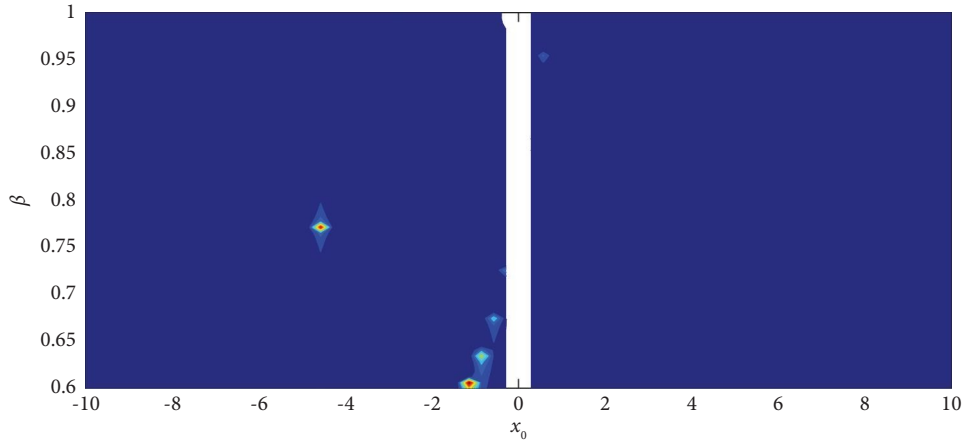


Figure 8: Convergence planes of second-order NR on $f_2(x)$ for Riemann–Liouville derivative.

It can be observed from Tables 1 and 2 that the estimation of second-order FNR is better than the one of the first-order FNR when the order of the derivative is close to one in both Caputo and Riemann–Liouville derivatives. In Figures 1–4, the convergence plane of the polynomial function $f_1(x)$ is given when $x_0 \in [-3, 3]$ for various values of β.

Example 2. Let us consider the function $f_2(x) = \exp(x) - 1$, whose only root is $x_1 = 0$.

It can be observed from Tables 3 and 4 that the estimation of second-order FNR is better than the one of the first-order FNR when the order of the derivative is close to one in both Caputo and Riemann–Liouville derivatives. In Figures 5–8, the convergence plane of the polynomial function $f_2(x)$ is given when $x_0 \in [-10, 10]$ for various values of β.

6. Conclusion

First- and second-order FNR are developed, and analyzed and applied in this study. Moreover, the convergence of both methods is established. It is shown that second-order FNR gives better results compare to first-order FNR when the order of fractional derivative is close to one in both Caputo and Riemann–Liouville derivatives. It is also shown that the order of convergence for first-order FNR is quadratic while one of the second-order FNR is 3/2. It is clear from tables that as the fractional parameter increases to one, the number of iterations decreases for both developed methods. Moreover, figures depict that the convergence of approximate solutions is better for $\beta \in (0.7, 1]$ which can be seen also from the tables. Generally, it is obvious from the obtained formulation that the order of convergence for k^{th} order FNR is $(k + 1/k)$. The obtained results are verified by presented examples, too.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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