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q-Analogs of $H^r_{n,m}(\sigma)$ and Their Applications

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Abstract: In this paper, inspired by recent works, we define q-analogs of $H_{n,m}(\sigma)$ and $H^r_{n,m}(\sigma)$. By implementing them, we obtain new interesting results by taking the derivative or using generating functions.

Keywords: alternative q-harmonic numbers; q-polylogarithms; generalized q-hyperharmonic numbers of order r

MSC: 05A15; 11G55; 11D68

1. Introduction

The Riemann zeta is stated by

$$\varsigma(m) = \sum_{n=0}^{\infty} \frac{1}{n^m} = \prod_{p\ prime} (1 - p^{-m})^{-1},$$

where m is a complex variant with $\mathrm{Re}(m) > 1$. For example, for $m = 1$, $\varsigma(1) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$, which may be shown to diverge, and for $m = 2$, the renowned fixed function [1,2] is

$$\varsigma(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi}{6}.$$

This function was first used in complex analysis in 1859 and is particularly important because of its remarkable connection with other branches of mathematics, like combinatoric and matrix theory, through the Riemann hypothesis.

Although it is the principal content in many of the basic perceptions in mathematics, there is no proof of the hypothesis yet.

To some extent, the Riemann zeta function is an amazing instrument that has attracted both mathematicians and physicists and also covers many different areas in these disciplines in spite of the fact that we do not know the full meaning of it.

For $m \in \mathbb{Z}$, the mth polylogarithm function is stated by a power series in t,

$$Li_m(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^m},$$

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which is convergent for $|t| < 1$. The harmonic numbers, indicated by H_n, are expressed by

$$H_0 = 0 \text{ and } H_n = \sum_{k=1}^{n} \frac{1}{k} \text{ for } n \geq 1,$$

and their generating function is

$$\frac{-\ln(1-t)}{1-t} = \sum_{n=0}^{\infty} H_n t^n.$$

For a long time, authors have studied harmonic numbers, which are very significant in miscellaneous branches of number theory. In different research areas, harmonic numbers have been considered [3–6]. There are a lot of works involving harmonic numbers and generalizations of them [7–13]. For brief explanations of these numbers, the readers can refer to WolframMathWorld's website [14].

Guo and Cha [15] defined the generalized harmonic numbers by

$$H_0(\sigma) = 0 \text{ and } H_n(\sigma) = \sum_{k=1}^{n} \frac{\sigma^k}{k} \text{ for } n \geq 1,$$

where σ is an appropriate parameter, and their generating function is

$$\frac{-\ln(1-\sigma t)}{1-t} = \sum_{n=0}^{\infty} H_n(\sigma) t^n.$$

When $\sigma = 1/\alpha$ for $\alpha \in \mathbb{R}^+$, $H_n(1/\alpha) := \sum_{k=1}^{n} \frac{1}{k\alpha^k}$ were called the generalized harmonic numbers by Genčev [16].

A q-analog or q-extension is a mathematical expansion parameterized by a quantity q that generalizes a known expression. It reduces to the known expression in the limit $q \to 1$. The terms in q-calculus were first defined and named by Euler in the 18th century. Thereafter, scientists gave some applications of q-analogs, such as q-hypergeometric functions, q-identities and q-Bessel functions, in the 19th century. Many applications have been presented for q-analogs in the 20th century. Jackson introduced the theory of q-calculus [10,16]. Works still continue today. q-calculus is an extension of basic calculus and many authors have examined it in different branches of science and engineering. There are a lot of applications in different mathematical areas such as special polynomials, number theory, combinatoric, hypergeometric functions and quantum mechanics [11,17–22].

For $k \in \mathbb{N}$, the q-analog of k is given by

$$k_q = 1 + q + q^2 + \ldots + q^{k-1}.$$

Throughout this paper, we assume q to be a fixed number satisfying $0 < q < 1$. The q-factorial of k_q is expressed by

$$0_q! = 1 \text{ and } k_q! = \prod_{i=1}^{k} i_q.$$

It is immediately observed that $\lim_{q \to 1^-} k_q! = k!$.

Clearly,

$$(t;q)_n = \begin{cases} \prod_{i=0}^{n-1} (1 - tq^i), & \text{if } n \geq 1, \\ 1, & \text{if } n = 0, \end{cases}$$

which is called the q-Pochhammer symbol.

For any $s, r \in \mathbb{N}^* = \{0, 1, 2, \ldots\}$, widely recognized q-binomial coefficients, also referred as the Gaussian coefficients, are given by the expression

$$\binom{s}{r}_q = \begin{cases} \frac{(q;q)_s}{(q;q)_r (q;q)_{s-r}}, & \text{if } s \geq r, \\ 0, & \text{if } s < r. \end{cases}$$

For example,

$$\binom{4}{2}_q = \left(q^2+1\right)\left(q^2+q+1\right).$$

Clearly,

$$\lim_{q \to 1^-} \binom{s}{r}_q = \binom{s}{r},$$

where $\binom{s}{r}$ is the usual binomial coefficient.

Recently, various features for sums involving q-polylogarithm functions have been introduced, see [23,24].

For $k \in \mathbb{Z}$, the kth q-polylogarithm function is expressed as follows:

$$Li_{q;k}(t) = \sum_{n=1}^{\infty} \frac{t^n}{n_q^k}. \tag{1}$$

Note that $\lim_{q \to 1^-} Li_{q;k}(t) = Li_k(t)$. For $k \leq 1$, the q-polylogarithm function [24] is shown by a rational function

$$Li_{q;k}(t) = (1-q)^{-k} \sum_{n=1}^{\infty} (-1)^n \frac{q^n t}{1-q^n t} \binom{k}{n}.$$

In [25], Rothe's formula is obtained by

$$\sum_{j=0}^{r} (-1)^j q^{\binom{j}{2}} \binom{r}{j}_q t^j = (t;q)_r, \tag{2}$$

and also it is known that Heine's binomial formula is denoted as

$$\frac{1}{(t;q)_r} = \sum_{j=0}^{\infty} \binom{r+j-1}{j}_q t^j. \tag{3}$$

In [26,27], the q-exponential function is expressed by

$$e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{n_q!}, \; t, q \in \mathbb{C} \text{ with } |t| < 1. \tag{4}$$

In [28], the q-Stirling numbers of the second kind are shown with

$$\frac{1}{l_q!} \left(e_q(t) - 1\right)^l = \sum_{n=l}^{\infty} S_{2,q}(n,l) \frac{t^n}{n_q!}. \tag{5}$$

One of the most interesting topics is the q-derivative equation which was discovered by Jackson in [10,16,29,30]. He came up with the q-derivative of a function $f(t)$ as follows:

$$D_q f(t) = \left(\frac{d}{dt}\right)_q f(t) = \frac{f(qt) - f(t)}{qt - t} \text{ for } t \neq 0. \tag{6}$$

The q-derivative is known as the Jackson derivative. Clearly,

$$\lim_{q\to 1^-} D_q f(t) = Df(t),$$

in which D is the ordinary derivative operator. The q-derivative quotient rule is given by

$$D_q\left(\frac{f(t)}{g(t)}\right) = \frac{g(t)D_q f(t) - f(t)D_q g(t)}{g(qt)g(t)}, \ \ g(qt)g(t) \neq 0. \tag{7}$$

2. q-Analogue of $H^r_{n,m}(\sigma)$ and Their Applications

In this part, first, q-analogs of $H_{n,m}(\sigma)$ and $H^r_{n,m}(\sigma)$ are defined and then some results are given.

Definition 1. *For $n, m \in \mathbb{Z}^+$, alternative q-harmonic numbers are stated by*

$$H_{n,m}(q,\sigma) := \sum_{k=1}^{n} \frac{\sigma^k}{k_q^m}. \tag{8}$$

Then, using $H_{n,m}(q,\sigma)$, generalized q-hyperharmonic numbers of order r are stated as follows.

Definition 2. *For $m \in \mathbb{Z}^+$, generalized q-hyperharmonic numbers of order r, $H^r_{n,m}(q,\sigma)$ are defined by*

$$H^r_{n,m}(q,\sigma) = \begin{cases} \sum\limits_{i=1}^{n} q^i H^{r-1}_{i,m}(q,\sigma), & \text{if } n, r \geq 1, \\ 0, & \text{if } r < 0 \text{ or } n \leq 0, \end{cases} \tag{9}$$

where $H^0_{n,m}(q,\sigma) = q^{-n}\frac{\sigma^n}{n_q^m}$.

There is the following relationship between these numbers and $H_n(k;q)$. In particular, taking $d = 1$ in Definition 1 (see [8]) and $\sigma = q^{k-1}$ in (8), then

$$H_n(k;q) = \sum_{j=1}^{n} q^j H_{j,k}\left(q, q^{k-1}\right) = H^2_{n,k}\left(q, q^{k-1}\right).$$

It is obtained that

$$H^r_{n,m}(q,\sigma) = q^n H^{r-1}_{n,m}(q,\sigma) + H^r_{n-1,m}(q,\sigma).$$

Theorem 1. *For $m, r \in \mathbb{Z}^+$, it holds that*

$$\sum_{n=1}^{\infty} H^r_{n,m}(q,\sigma)t^n = \frac{1}{(t;q)_r} Li_{q;m}\left(q^{r-1}\sigma t\right). \tag{10}$$

Proof. By (9), observe that

$$\begin{aligned}\sum_{n=0}^{\infty} H_{n,m}^{r}(q,\sigma)t^n &= \sum_{n=0}^{\infty}\sum_{i=0}^{n} q^i H_{i,m}^{r-1}(q,\sigma)t^n = \frac{1}{1-t}\sum_{i=0}^{\infty} q^i H_{i,m}^{r-1}(q,\sigma)t^i \\ &= \frac{1}{1-t}\sum_{i=0}^{\infty} q^i \sum_{j=0}^{i} q^j H_{j,m}^{r-2}(q,\sigma)t^i \\ &= \frac{1}{1-t}\frac{1}{1-qt}\sum_{j=0}^{\infty} q^{2j} H_{j,m}^{r-2}(q,\sigma)t^j \\ &\cdots \\ &= \frac{1}{(t;q)_r}\sum_{j=0}^{\infty} q^{rj} H_{j,m}^{0}(q,\sigma)t^j = \frac{1}{(t;q)_r}\sum_{j=1}^{\infty} q^{rj} H_{j,m}^{0}(q,\sigma)t^j \\ &= \frac{1}{(t;q)_r}\sum_{j=1}^{\infty} q^{(r-1)j}\frac{\sigma^j}{j_q^m}t^j = \frac{1}{(t;q)_r} Li_{q;m}\left(q^{r-1}\sigma t\right).\end{aligned}$$

Thus, the proof of (10) is obtained. □

When $r = 1$ in Theorem 1, clearly

$$\sum_{n=1}^{\infty} H_{n,m}(q,\sigma)t^n = \frac{1}{1-t} Li_{q;m}(\sigma t).$$

Theorem 2. *For $m, n, r \in \mathbb{Z}^+$, it holds that*

$$H_{n,m}^{r}(q,\sigma) = \sum_{i=1}^{n} \frac{\sigma^i q^{(r-1)i}}{i_q^m}\binom{n-i+r-1}{r-1}_q.$$

Proof. By using (1), (3) and (10), we write

$$\begin{aligned}\sum_{n=1}^{\infty} H_{n,m}^{r}(q,\sigma)t^n &= \frac{1}{(t;q)_r} Li_{q;m}\left(q^{r-1}\sigma t\right) = \sum_{l=0}^{\infty}\binom{r+l-1}{l}_q t^l \sum_{j=1}^{\infty}\frac{\sigma^j q^{(r-1)j}}{j_q^m}t^j \\ &= \sum_{n=1}^{\infty}\sum_{i=1}^{n}\binom{n-i+r-1}{r-1}_q \frac{\sigma^i q^{(r-1)i}}{i_q^m}t^n.\end{aligned}$$

Thus, comparing the coefficients on the left and right sides, the result is obtained. □

Theorem 3. *For $m, n, r \in \mathbb{Z}^+$, it holds that*

$$\sum_{k=1}^{n}(-1)^k\binom{r}{n-k}_q q^{\binom{n-k}{2}} H_{k,m}^{r}(q,\sigma) = (-1)^n \frac{q^{(r-1)n}\sigma^n}{n_q^m}.$$

Proof. From (1), (2) and (10), we write

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{q^{(r-1)n}\sigma^n}{n_q^m}t^n = Li_{q;m}\left(q^{r-1}\sigma t\right) = (t;q)_r\sum_{n=1}^{\infty} H_{n,m}^{r}(q,\sigma)t^n \\ =\ &\sum_{k=0}^{\infty}(-1)^k q^{\binom{k}{2}}\binom{r}{k}_q t^k \sum_{n=1}^{\infty} H_{n,m}^{r}(q,\sigma)t^n \\ =\ &\sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{n-k} q^{\binom{n-k}{2}}\binom{r}{n-k}_q H_{k,m}^{r}(q,\sigma)t^n,\end{aligned}$$

as claimed. □

Theorem 4. *For $m, r, l, p \in \mathbb{Z}^+$ such that $m + l > p$, we have*

$$\begin{aligned}
&\sum_{i=1}^{n} i_q! H_{i,m-p+l}^{r+l}(q,\sigma) S_{2,q}(n,i) \\
&= \sum_{j=1}^{n} \sum_{k=0}^{n-j} \sum_{i=1}^{j} i_q! k_q! \frac{q^{(r+l-1)i}\sigma^i}{i_q^{m-p+l}} \binom{k+r+l-1}{k}_q \binom{n}{j}_q \\
&\qquad \times S_{2,q}(n-j,k) S_{2,q}(j,i).
\end{aligned}$$

Proof. Note that from (10),

$$\sum_{n=0}^{\infty} H_{n,m-p+l}^{r+l}(q,\sigma) t^n = \frac{Li_{q;m-p+l}\left(q^{r+l-1}\sigma t\right)}{(t;q)_{r+l}}.$$

Thus, (5) yields that

$$\begin{aligned}
&\frac{1}{(e_q(t)-1;q)_{r+l}} Li_{q;m-p+l}\left(q^{r+l-1}\sigma(e_q(t)-1)\right) \\
&= \sum_{i=1}^{\infty} i_q! H_{i,m-p+l}^{r+l}(q,\sigma) \frac{(e_q(t)-1)^i}{i_q!} \\
&= \sum_{i=1}^{\infty} i_q! H_{i,m-p+l}^{r+l}(q,\sigma) \sum_{n=i}^{\infty} S_{2,q}(n,i) \frac{t^n}{n_q!} \\
&= \sum_{n=1}^{\infty} \sum_{i=1}^{n} H_{i,m-p+l}^{r+l}(q,\sigma) S_{2,q}(n,i) \frac{i_q!}{n_q!} t^n. \qquad (11)
\end{aligned}$$

(1), (3) and (5) yield that

$$\begin{aligned}
&\frac{1}{(e_q(t)-1;q)_{r+l}} Li_{q;m-p+l}\left(q^{r+l-1}\sigma(e_q(t)-1)\right) \\
&= \sum_{k=0}^{\infty} \binom{k+r+l-1}{k}_q (e_q(t)-1)^k \sum_{i=1}^{\infty} \frac{q^{(r+l-1)i}\sigma^i}{i_q^{m-p+l}} (e_q(t)-1)^i \\
&= \sum_{k=0}^{\infty} k_q! \binom{k+r+l-1}{k}_q \sum_{n=k}^{\infty} S_{2,q}(n,k) \frac{t^n}{n_q!} \\
&\qquad \times \sum_{i=1}^{\infty} \frac{q^{(r+l-1)i}\sigma^i}{i_q^{m-p+l}} i_q! \sum_{n=i}^{\infty} S_{2,q}(n,i) \frac{t^n}{n_q!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{k_q!}{n_q!} \binom{k+r+l-1}{k}_q S_{2,q}(n,k) t^n \\
&\qquad \times \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{q^{(r+l-1)i}\sigma^i}{i_q^{m-p+l}} \frac{i_q!}{n_q!} S_{2,q}(n,i) t^n \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{n} \sum_{k=0}^{n-j} \sum_{i=1}^{j} \frac{q^{(r+l-1)i}\sigma^i}{i_q^{m-p+l}} \frac{i_q! k_q!}{n_q!} \binom{k+r+l-1}{k}_q \binom{n}{j}_q \\
&\qquad \times S_{2,q}(n-j,k) S_{2,q}(j,i) t^n. \qquad (12)
\end{aligned}$$

Hence, the result is obtained by equating the coefficients from (11) and (12). □

Lemma 1. *For $m, r \in \mathbb{Z}^+$, it holds that*

$$D_q((q^m t;q)_r) = -q^m r_q\left(q^{m+1}t;q\right)_{r-1},$$

and

$$D_q^n\left(Li_{q;m}\left(q^{r-1}\sigma t\right)\right) = n_q!\sum_{k=0}^{\infty}\binom{k+n}{n}_q \frac{q^{(r-1)(k+n)}\sigma^{k+n}}{(k+n)_q^m}t^k. \tag{13}$$

Proof. By (2) and (6), we write

$$\begin{aligned} D_q((q^m t;q)_r) &= D_q\left(\sum_{i=0}^{r}(-1)^i q^{\binom{i}{2}}\binom{r}{i}_q q^{mi}t^i\right) \\ &= \sum_{i=0}^{r-1}(-1)^{i+1}q^{\binom{i+1}{2}}\binom{r}{i+1}_q (i+1)_q q^{m(i+1)}t^i \\ &= -q^m r_q\sum_{i=0}^{r-1}(-1)^i q^{\binom{i}{2}}\binom{r-1}{i}_q\left(q^{m+1}t\right)^i \\ &= -q^m r_q\left(q^{m+1}t;q\right)_{r-1}. \end{aligned}$$

Using (1), (13) is similarly obtained. □

Theorem 5. *For $m,n,p,r \in \mathbb{Z}^+$, it holds that*

$$\sum_{l=0}^{n}\sum_{i=0}^{l}(-1)^{l-i}q^{\binom{l-i}{2}+n-rl}\binom{p+r}{l-i}_q\binom{p+n-l-1}{n-l}_q H_{i,m}^r(q,\sigma) = \frac{\sigma^n}{n_q^m}.$$

Proof. By $(t;q)_{k+l} = (t;q)_l\left(tq^l;q\right)_k$, we write

$$D_q^n\left(Li_{q;m}\left(q^{r-1}\sigma t\right)\right) = D_q^n\left(\frac{Li_{q;m}\left(q^{r-1}\sigma t\right)}{(t;q)_r}\frac{(t;q)_{p+r}}{(tq^r;q)_p}\right),$$

and from (2), (3) and (10),

$$\begin{aligned} & D_q^n\left(\sum_{n=0}^{\infty}H_{n,m}^r(q,\sigma)t^n\sum_{i=0}^{\infty}(-1)^i q^{\binom{i}{2}}\binom{r+k}{i}_q t^i\sum_{k=0}^{\infty}\binom{p+k-1}{k}_q q^{rk}t^k\right) \\ = & D_q^n\left(\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{i=0}^{l}(-1)^{l-i}q^{\binom{l-i}{2}+r(j-l)}\binom{p+r}{l-i}_q\binom{p+j-l-1}{j-l}_q H_{i,m}^r(q,\sigma)t^j\right) \\ = & D_q^{n-1}\left(\sum_{j=1}^{\infty}\sum_{l=0}^{j}\sum_{i=0}^{l}(-1)^{l-i}q^{\binom{l-i}{2}+r(j-l)}j_q\binom{p+r}{l-i}_q\binom{p+j-l-1}{j-l}_q H_{i,m}^r(q,\sigma)t^{j-1}\right) \\ = & \cdots \\ = & \sum_{j=n}^{\infty}\sum_{l=0}^{j}\sum_{i=0}^{l}(-1)^{l-i}q^{\binom{l-i}{2}+r(j-l)}n_q!\binom{p+r}{l-i}_q \\ & \times\binom{p+j-l-1}{j-l}_q\binom{j}{n}_q H_{i,m}^r(q,\sigma)t^{j-n} \\ = & \sum_{j=0}^{\infty}\sum_{l=0}^{n+j}\sum_{i=0}^{l}(-1)^{l-i}q^{\binom{l-i}{2}+r(n+j-l)}n_q!\binom{p+r}{l-i}_q \\ & \times\binom{p+j+n-l-1}{j+n-l}_q\binom{j+n}{n}_q H_{i,m}^r(q,\sigma)t^j. \end{aligned} \tag{14}$$

Thus, the result is obtained by equating the coefficients from (13) and (14). □

Theorem 6. *For $n, m, r, s \in \mathbb{Z}^+$, it holds that*

$$\begin{aligned}
&\frac{1}{s_q!}D_q^s\left(\frac{1}{(t;q)_r}Li_{q;m}\left(q^{r-1}\sigma t\right)\right)\\
=\;&\binom{r+s-1}{s}_q\frac{1}{(t;q)_{r+s}}Li_{q;m}\left(q^{r-1}\sigma t\right)\\
&+\sum_{n=0}^{\infty}\sum_{i=1}^{s}\binom{n+i}{i}_q\binom{r+s-i-1}{s-i}_q\frac{1}{(q^it;q)_{r+s-i}}\frac{q^{(r-1)(n+i)}\sigma^{n+i}}{(n+i)_q^m}t^n,
\end{aligned}$$

where D_q is given by (6).

Proof. With the help of (7) and Lemma 1, observe that

$$\begin{aligned}
&D_q\left(\frac{1}{(t;q)_r}Li_{q;m}\left(q^{r-1}\sigma t\right)\right)\\
=\;&\frac{(t;q)_rD_q\left(Li_{q;m}\left(q^{r-1}\sigma t\right)\right)-Li_{q;m}\left(q^{r-1}\sigma t\right)D_q\left((t;q)_r\right)}{(qt;q)_r(t;q)_r}\\
=\;&\frac{1}{(qt;q)_r}\sum_{n=0}^{\infty}\frac{q^{(r-1)(n+1)}\sigma^{n+1}}{(n+1)_q^{m-1}}t^n+r_q\frac{1}{(t;q)_{r+1}}Li_{q;m}\left(q^{r-1}\sigma t\right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&D_q^2\left(\frac{1}{(t;q)_r}Li_{q;m}\left(q^{r-1}\sigma t\right)\right)=\frac{1}{(q^2t;q)_r(qt;q)_r}\\
&\times\left((qt;q)_rD_q\left(\sum_{n=0}^{\infty}\frac{q^{(r-1)(n+1)}\sigma^{n+1}}{(n+1)_q^{m-1}}t^n\right)+qr_q\left(q^2t;q\right)_{r-1}\sum_{n=0}^{\infty}\frac{q^{(r-1)(n+1)}\sigma^{n+1}}{(n+1)_q^{m-1}}t^n\right)\\
&+\frac{r_q}{(qt;q)_{r+1}(t;q)_{r+1}}\left((t;q)_{r+1}\sum_{n=0}^{\infty}\frac{q^{(r-1)(n+1)}\sigma^{n+1}}{(n+1)_q^{m-1}}t^n+(r+1)_q(qt;q)_rLi_{q;m}\left(q^{r-1}\sigma t\right)\right)\\
=\;&\frac{1}{(q^2t;q)_r}\sum_{n=0}^{\infty}\frac{q^{(r-1)(n+2)}\sigma^{n+2}}{(n+2)_q^{m-1}}(n+1)_qt^n+(q+1)r_q\frac{1}{(qt;q)_{r+1}}\sum_{n=0}^{\infty}\frac{q^{(r-1)(n+1)}\sigma^{n+1}}{(n+1)_q^{m-1}}t^n\\
&+r_q(r+1)_q\frac{1}{(t;q)_{r+2}}Li_{q;m}\left(q^{r-1}\sigma t\right),
\end{aligned}$$

and

$$\begin{aligned}
&D_q^3\left(\frac{1}{(t;q)_r}Li_{q;m}\left(q^{r-1}\sigma t\right)\right)=\frac{1}{(q^3t;q)_r}\sum_{n=0}^{\infty}\frac{q^{(r-1)(n+3)}\sigma^{n+3}}{(n+3)_q^{m-1}}(n+2)_q(n+1)_qt^n\\
&+\frac{q^2r_q}{(q^2t;q)_{r+1}}\sum_{n=0}^{\infty}\frac{q^{(r-1)(n+2)}\sigma^{n+2}}{(n+2)_q^{m-1}}(n+1)_qt^n+\frac{(q+1)r_q}{(q^2t;q)_{r+1}}\sum_{n=0}^{\infty}\frac{q^{(r-1)(n+2)}\sigma^{n+2}}{(n+2)_q^{m-1}}(n+1)_qt^n\\
&+\frac{q(q+1)r_q(r+1)_q}{(qt;q)_{r+2}}\sum_{n=0}^{\infty}\frac{q^{(r-1)(n+1)}\sigma^{n+1}}{(n+1)_q^{m-1}}t^n+\frac{r_q(r+1)_q}{(qt;q)_{r+2}}\sum_{n=0}^{\infty}\frac{q^{(r-1)(n+1)}\sigma^{n+1}}{(n+1)_q^{m-1}}t^n\\
&+\frac{r_q(r+1)_q(r+2)_q}{(t;q)_{r+3}}Li_{q;m}\left(q^{r-1}\sigma t\right)\\
=\;&\frac{(n+2)_q(n+1)_q}{(q^3t;q)_r}\sum_{n=0}^{\infty}\frac{q^{(r-1)(n+3)}\sigma^{n+3}}{(n+3)_q^{m-1}}t^n+\frac{3_qr_q}{(q^2t;q)_{r+1}}\sum_{n=0}^{\infty}\frac{q^{(r-1)(n+2)}\sigma^{n+2}}{(n+2)_q^{m-1}}(n+1)_qt^n\\
&+\frac{3_qr_q(r+1)_q}{(qt;q)_{r+2}}\sum_{n=0}^{\infty}\frac{q^{(r-1)(n+1)}\sigma^{n+1}}{(n+1)_q^{m-1}}t^n+\frac{r_q(r+1)_q(r+2)_q}{(t;q)_{r+3}}Li_{q;m}\left(q^{r-1}\sigma t\right).
\end{aligned}$$

Continuing this process, the proof is complete. □

Corollary 1. *For $n, m, s, r \in \mathbb{Z}^+$, it holds that*

$$\begin{aligned} & H^r_{n+s,m}(q,\sigma)\begin{pmatrix} n+s \\ s \end{pmatrix}_q \\ = & \begin{pmatrix} r+s-1 \\ s \end{pmatrix}_q \sum_{j=0}^{n} q^{r(n-j)} \begin{pmatrix} n+s-j-1 \\ s-1 \end{pmatrix}_q H^r_{j,m}(q,\sigma) \\ & + \sum_{j=0}^{n}\sum_{i=1}^{s} \begin{pmatrix} r+s-i+j-1 \\ j \end{pmatrix}_q \begin{pmatrix} n-j+i \\ i \end{pmatrix}_q \begin{pmatrix} r+s-i-1 \\ s-i \end{pmatrix}_q \frac{q^{(r-1)(n-j+i)} q^{ij} \sigma^{n-j+i}}{(n-j+i)_q^m}. \end{aligned}$$

Proof. Observe that

$$\begin{aligned} & D_q^s\left(\frac{1}{(t;q)_r} Li_{q;m}\left(q^{r-1}\sigma t\right)\right) \\ = & D_q^{s-1}\left(\sum_{n=0}^{\infty} H^r_{n+1,m}(q,\sigma)(n+1)_q t^n\right) \\ = & \dots \\ = & \sum_{n=0}^{\infty} H^r_{n+s,m}(q,\sigma)\begin{pmatrix} n+s \\ s \end{pmatrix}_q s_q! t^n. \end{aligned} \tag{15}$$

From Theorem 6 and $(t;q)_{r+s} = (t;q)_r (tq^r;q)_s$, we write

$$\begin{aligned} & \begin{pmatrix} r+s-1 \\ s \end{pmatrix}_q \frac{1}{(t;q)_{r+s}} Li_{q;m}\left(q^{r-1}\sigma t\right) \\ & + \sum_{n=0}^{\infty}\sum_{i=1}^{s} \begin{pmatrix} n+i \\ i \end{pmatrix}_q \begin{pmatrix} r+s-i-1 \\ s-i \end{pmatrix}_q \frac{1}{(q^i t;q)_{r+s-i}} \frac{q^{(r-1)(n+i)}\sigma^{n+i}}{(n+i)_q^m} t^n \\ = & \begin{pmatrix} r+s-1 \\ s \end{pmatrix}_q \frac{1}{(tq^r;q)_s} \frac{1}{(t;q)_r} Li_{q;m}\left(q^{r-1}\sigma t\right) \\ & + \sum_{n=0}^{\infty}\sum_{i=1}^{s} \begin{pmatrix} n+i \\ i \end{pmatrix}_q \begin{pmatrix} r+s-i-1 \\ s-i \end{pmatrix}_q \frac{1}{(q^i t;q)_{r+s-i}} \frac{q^{(r-1)(n+i)}\sigma^{n+i}}{(n+i)_q^m} t^n, \end{aligned}$$

and by (3) and (10), using the product of two series, this equals

$$\begin{aligned} & \begin{pmatrix} r+s-1 \\ s \end{pmatrix}_q \sum_{n=0}^{\infty}\sum_{j=0}^{n} \begin{pmatrix} s+j-1 \\ j \end{pmatrix}_q q^{rj} H^r_{n-j,m}(q,\sigma) t^n \\ & + \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{i=1}^{s} \begin{pmatrix} r+s-i+j-1 \\ j \end{pmatrix}_q \begin{pmatrix} n-j+i \\ i \end{pmatrix}_q \begin{pmatrix} r+s-i-1 \\ s-i \end{pmatrix}_q \frac{q^{(r-1)(n-j+i)} q^{ij} \sigma^{n-j+i}}{(n-j+i)_q^m} t^n. \end{aligned} \tag{16}$$

Hence, the result is obtained by equating the coefficients from (15) and (16). □

For example, when $r = s = 1$ in Corollary 1,

$$\sum_{j=0}^{n} q^{-j} H_{j,m}(q,\sigma) = q^{-n}(n+1)_q H_{n+1,m}(q,\sigma) - qH_{n+1,m-1}\left(q, q^{-1}\sigma\right).$$

Furthermore, when $n = r = 1$ in Corollary 1, writing n instead of s,

$$\sum_{i=1}^{n} (n-i+1)_q \frac{q^i \sigma^i}{i_q^m} = (n+1)_q H_{n+1,m}(q,\sigma) - H_{n+1,m-1}(q,\sigma).$$

Theorem 7. *For $n, m, j, r \in \mathbb{Z}^+$, it holds that*

$$\begin{aligned}
&\sum_{l=1}^{n}\sum_{i=1}^{l} i_q! \binom{n}{l}_q H_{i,m-k+j}^{r+j}(q,\sigma) S_{2,q}(l,i) \\
&= \sum_{u=1}^{n}\sum_{l=1}^{u}\sum_{p=0}^{u-l}\sum_{i=1}^{l} \frac{q^{(r+j-1)i}\sigma^i}{i_q^{m-k+j}} i_q! p_q! \binom{p+r+j-1}{p}_q \binom{n}{u}_q \binom{u}{l}_q \\
&\qquad \times S_{2,q}(u-l,p) S_{2,q}(l,i).
\end{aligned}$$

Proof. From (4) and (11), observe that

$$\begin{aligned}
&\frac{1}{(e_q(t)-1;q)_{r+j}} Li_{q;m-k+j}\left(q^{r+j-1}\sigma\left(e_q(t)-1\right)\right) e_q(t) \\
&= \sum_{i=1}^{\infty} H_{i,m-k+j}^{r+j}(q,\sigma) i_q! \sum_{n=i}^{\infty} \frac{S_{2,q}(n,i)}{n_q!} t^n \sum_{n=0}^{\infty} \frac{t^n}{n_q!} \\
&= \sum_{n=1}^{\infty}\sum_{l=1}^{n}\sum_{i=1}^{l} H_{i,m-k+j}^{r+j}(q,\sigma) S_{2,q}(l,i) \frac{i_q!}{l_q!} \frac{1}{(n-l)_q!} t^n \\
&= \sum_{n=1}^{\infty}\sum_{l=1}^{n}\sum_{i=1}^{l} H_{i,m-k+j}^{r+j}(q,\sigma) S_{2,q}(l,i) \frac{i_q!}{n_q!} \binom{n}{l}_q t^n. \qquad (17)
\end{aligned}$$

(4) and (12) yield that

$$\begin{aligned}
&\frac{1}{(e_q(t)-1;q)_{r+j}} Li_{q;m-k+j}\left(q^{r+j-1}\sigma\left(e_q(t)-1\right)\right) e_q(t) \\
&= \sum_{n=1}^{\infty}\sum_{l=1}^{n}\sum_{p=0}^{n-l}\sum_{i=1}^{l} \frac{q^{(r+j-1)i}\sigma^i}{i_q^{m-k+j}} \frac{i_q! p_q!}{n_q!} \binom{p+r+j-1}{p}_q \binom{n}{l}_q \\
&\qquad \times S_{2,q}(n-l,p) S_{2,q}(l,i) t^n \sum_{n=0}^{\infty} \frac{t^n}{n_q!} \\
&= \sum_{n=1}^{\infty}\sum_{u=1}^{n}\sum_{l=1}^{u}\sum_{p=0}^{u-l}\sum_{i=1}^{l} \frac{q^{(r+j-1)i}\sigma^i}{i_q^{m-k+j}} \frac{i_q! p_q!}{n_q!} \binom{p+r+j-1}{p}_q \binom{n}{u}_q \binom{u}{l}_q \\
&\qquad \times S_{2,q}(u-l,p) S_{2,q}(l,i) t^n. \qquad (18)
\end{aligned}$$

Thus, the result is obtained by equating the coefficients from (17) and (18). □

3. Conclusions

In this paper, as a further generalization of hyperharmonic numbers, we defined q-analogs of generalized hyperharmonic numbers of order r, $H_{n,m}^r(q,\sigma)$, and gave the generating functions of $H_{n,m}^r(q,\sigma)$ in Theorem 1 and the closed form of $H_{n,m}^r(q,\sigma)$ in Theorem 2. Features including the q-Stirling numbers of the second kind in Theorems 4 and 7 are obtained. Furthermore, the applications of the derivation are considered in Theorems 5 and 6. In future studies, the following sum can be considered as a generalization of the numbers mentioned in the paper, and interesting equalities containing these new numbers can be given by

$$\sum_{k=1}^{n} \frac{\sigma^k}{(k+x)_q}$$

for $x \in \mathbb{R}^+$. Furthermore, one fascinating aspect of applying combinatorial theorems in mathematics is that they simply express multiple sums. One of our future projects is to find some of their possible implementations in mathematics, science and engineering. Further progress on this work will be presented in future research.

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