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Generalization of quantum calculus and corresponding Hermite–Hadamard inequalities

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Abstract
The aim of this paper is first to introduce generalizations of quantum integrals and derivatives which are called $(\phi - h)$ integrals and $(\phi - h)$ derivatives, respectively. Then we investigate some implicit integral inequalities for $(\phi - h)$ integrals. Different classes of convex functions are used to prove these inequalities for symmetric functions. Under certain assumptions, Hermite–Hadamard-type inequalities for q-integrals are deduced. The results presented herein are applicable to convex, m-convex, and $\hbar$-convex functions defined on the non-negative part of the real line.

Keywords $(\phi - h)$-derivative · $(\phi - h)$-integral · Jensen inequality · $\hbar$-convex function · m-convex function · Hermite Hadamard Inequality

Mathematics Subject Classification 26A51 · 26A33 · 33E12

1 Introduction

Theory of integral inequalities provides useful tools to estimate an integral of functions over intervals. These inequalities give bounds to estimate the cumulative behavior of functions over a specified domain and provide insight into the general behavior of different mathematical objects. On the other hand, Quantum calculus known as q-Calculus is based on the idea of studying the concepts of classical calculus without

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taking into accounts the limits. In the nineteenth century, Euler [14] investigated several notable results such as Jacobi's triple product identity and theory of q-hypergeometric functions. The study of quantum integral inequalities has gained much attention of researchers these days. The concept of the definite q-integral was first introduced by Jackson [13] in 1910. Indeed, he was the first who developed q-calculus in a systematic way. In, 2013 and 2014, Tariboon and Ntouyas [23, 24] introduced the concepts of q-derivatives and q-integrals over finite intervals and derived quantum integral inequalities. In 2018, Alp et al. [3] employed the left q-integral operator to develop q-Hermite–Hadamard inequality using convex and quasi-convex functions. Also, Alomari [2] derived the quantum variant of Bernoulli inequality. In 2022, Liu et al. [15] proved the q-Hermite–Hadamard inequality by using $(q-h)$ -derivatives and integrals. Recently, Chen et al. [7] derived some results for $(q-h)$-integrals using $\hbar$-convex and m-convex functions and Farid et al. [9] proved the generalized q-integral inequalities using $(\hbar-m)$ and $(\alpha-m)$ convexities. For some other recent developments in this direction, we refer to [4–6, 8, 10, 11, 14, 16, 17, 22] and references mentioned therein.

In this paper, a new version of Hermite–Hadamard quantum integral inequalities for convex, m-convex and $\hbar$-convex functions using the left and right $(\phi-h)$-integrals are obtained. The results of this paper can be applied to convex functions defined on non negative part of real line $\mathbb{R}$.

Definition 1 [20] If I is any interval in $\mathbb{R}$. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex function if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \forall x, y \in I \text{ and } \lambda \in [0,1]. \quad (1.1)$$

If the inequality is reversed, then f is concave function.

Definition 2 [26] Suppose that I and Ω are any two intervals in $\mathbb{R}$ and $\hbar : (0,1) \subseteq \Omega \to \mathbb{R}$ is a nonnegative function. A nonnegative function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be $\hbar$-convex if

$$f(\lambda x + (1-\lambda)y) \leq \hbar(\lambda) f(x) + \hbar(1-\lambda) f(y), \forall x, y \in I \text{ and } \lambda \in [0,1] \text{ holds.} \quad (1.2)$$

If the inequality is reversed then f is $\hbar$-concave. If $\hbar(\lambda) = \lambda$ in (1.2), then the above inequality reduces to (1.1), and if we take, $\hbar(\lambda) = \lambda^s$ in (1.2), we get the definition of s-convex function given in [12].

Definition 3 [25] Let $c > 0$ and $0 \leq m \leq 1$. The function $f : [0, c] \to \mathbb{R}$ is called m-convex, if the following inequality holds

$$f(\lambda x + m(1-\lambda)y) \leq \lambda f(x) + m(1-\lambda) f(y), \forall x, y \in [0, c] \text{ and } \lambda \in [0,1]. \quad (1.3)$$

If $m = 1$ in (1.3), then we get (1.1). If we take, $m = 0$, then we get a star shaped function. That is, for any point $x \in [0, c]$ and $t \in [0,1]$ we have $f(tx) \leq t f(x)$.

Theorem 1 [19] *If $f : [a, b] \to \mathbb{R}$ is a convex function, where $a < b$ and $[a, b] \subset \mathbb{R}$, then, we have the Hermite–Hadamard inequality which is given as follows:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.4}$$

Definition 4 [20] Let f be a convex function defined on an open interval (a, b), $x_i \in (a, b)$ and $\alpha_i > 0$ with $\sum_{i=1}^{n} \alpha_i = 1$. Then, the following inequality

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i f(x_i), \tag{1.5}$$

is called the Jensen inequality and the function f is known as a J -convex function. The following result is an extension of Hermite–Hadamard inequality to the $\hbar$-convex function [21].

Theorem 2 *If $f : I \to \mathbb{R}$ is a $\hbar$-convex function on the interval $I \subset \mathbb{R}$ and $a, b \in I$ where $a < b$. Then, the generalized form of Hermite–Hadamard inequality is given as follows:*

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq (f(a)+f(b))\int_0^1 \hbar(t)dt. \tag{1.6}$$

In an ordinary calculus, the classical derivative of a function f is defined as follows:

$$\frac{d}{dx} f(x) = \lim_{x \to a} \frac{f(a) - f(x)}{a - x}, \text{ provided that the limit exists.} \tag{1.7}$$

Quantum derivatives without involving limit have been defined in various ways.
We present two types of derivatives known as q-derivative and the h -derivative for a given function f.
Let us recall the notations of a quantum differential given as.

Definition 5 [14] The q-differential is:

$$d_q f(x) = f(qx) - f(x), \text{ whereas the } h\text{-differential is: } d_h f(x) = f(x+h) - f(x). \tag{1.8}$$

In particular, $d_q x = (q-1)x$ and $d_h x = h$. These two quantum differentials allow us to define the corresponding quantum derivatives.

Definition 6 [14] For a continuous function $f : I \to \mathbb{R}$,

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x}, \text{ and} \tag{1.9}$$

$$D_h f(x) = \frac{d_h f(x)}{d_h x} = \frac{f(x+h) - f(x)}{h}, \tag{1.10}$$

are called the q-derivative and h-derivative, respectively. Note that: If f is differentiable, then we have $\lim_{q\to 1} D_q f(x) = \lim_{h\to 0} D_h f(x) = \frac{df(x)}{dx}$.

Definition 7 [10] For a continuous function $f : I \to \mathbb{R}$, the $(q-h)$-derivative is expressed as:

$$D_q^h f(x) = \frac{d_q^h f(x)}{d_q^h x} = \frac{f(q(x+h)) - f(x)}{(q-1)x + qh}, \; x_0 := \frac{qh}{1-q} \neq x \tag{1.11}$$

where $q \in (0,1)$ and $h \in \mathbb{R}$. If $x = x_0$, then we have $D_q^h f(x_0) = \lim_{x\to x_0} D_q^h f(x)$.

When $h = 0$ and $q = 1$ in (7), we get quantum derivative and plank's derivative, respectively defined in (6).
The Jackson integral or q-integral according to [13], is expressed as:

Definition 8 For a continuous function f, $0 < q < 1$ the q-integral on $[0, a]$ is expressed as:

$$\int_0^a f(x) d_q x = \sum_{n=0}^{\infty} a(1-q) q^n f(q^n a). \tag{1.12}$$

Note that,

$$\lim_{q\to 1} \int_0^a f(x) d_q x = \int_0^a f(x) dx. \tag{1.13}$$

If $c \in (0, b)$, then q-integral of f is defined as:

$$\int_c^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^c f(x) d_q x. \tag{1.14}$$

Definition 9 [24] Assume that $f : [a, b] \to \mathbb{R}$ is a continuous function, $x \in [a, b]$, and $0 < q < 1$. Then q_a-definite integral on $[a, b]$ is defined as:

$$\int_a^x f(t)_a dt_q = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)x), x > a.$$

Also, q^b-definite integral on $[a, b]$ is defined as:

$$\int_x^b f(t)^b dt_q = (1-q)(b-x) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b), x < b.$$

When $a = 0$ in (9), we get Jackson integral defined in (8).

Definition 10 [15] Let $f : [a, b] \to \mathbb{R}$ be a continuous function, $x \in [a, b], 0 < q < 1$ and $h \in \mathbb{R}$. Then $(q_a - h)$-integral on $[a, b]$ is defined as:

$$\int_a^x f(t)_h d_q^a t = ((1-q)(x-a) + qh) \sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)x + nq^n h), x > a,$$

whereas, $(q^b - h)$-integral on $[a, b]$ is defined as:

$$\int_x^b f(t)_h d_q^b t = ((1-q)(b-x) + qh) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b + nq^n h), x < b.$$

When $h = 0$ in Definition (10), we get integrals defined in (9).
Let us now recall the results on quantum Hermite–Hadamard integral inequalities:

Theorem 3 *[3] Let $0 < q < 1$, and $f : [a, b] \to \mathbb{R}$ a differentiable convex function on an interval (a, b). Then, the q -Hermite–Hadamard inequality on q_a-integral is given as:*

$$f\left(\frac{aq+b}{1+q}\right) \le \frac{1}{b-a}\int_a^b f(x) d_q^a x \le \frac{qf(a)+f(b)}{1+q}. \tag{1.15}$$

Theorem 4 [3] *Let $0 < q < 1$, and $f : [a, b] \to \mathbb{R}$ a differentiable convex function on an interval (a, b), the q-Hermite–Hadamard inequality on q_a-integral is given by:*

$$f\left(\frac{a+bq}{1+q}\right) + f'\left(\frac{a+bq}{1+q}\right)\frac{(b-a)(1-q)}{1+q} \le \frac{1}{b-a}\int_a^b f(x) d_q^a x \le \frac{qf(a)+f(b)}{1+q}. \tag{1.16}$$

Theorem 5 [3] *Let $0 < q < 1$, and $f : [a, b] \to \mathbb{R}$ a differentiable convex function on (a, b), the q-Hermite–Hadamard inequality on q_a -integral is given as:*

$$f\left(\frac{a+b}{1+q}\right) + f'\left(\frac{a+b}{1+q}\right)\frac{(b-a)(1-q)}{2(1+q)} \le \frac{1}{b-a}\int_a^b f(x) d_q^a x \le \frac{qf(a)+f(b)}{1+q}. \tag{1.17}$$

Furthermore, the q-Hadamrd inequality for convex functions was established for q_b-integral.

Theorem 6 [4] *Let $f : [a, b] \to \mathbb{R}$ be a differentiable convex function on an interval (a, b). Then, the q-Hadamard inequality on q_b -integral is given as:*

$$f\left(\frac{a+bq}{1+q}\right) \le \frac{1}{b-a}\int_a^b f(x) d_q^b x \le \frac{f(a)+qf(b)}{1+q}. \tag{1.18}$$

The aforementioned quantum integral inequalities have further generalized forms (see [1, 15, 18, 24]).

2 A genaralization of qauntum calculus

In this section, we define $(\phi - h)$-derivative and $(\phi - h)$-integral.

Definition 11 Let $f : I \to \mathbb{R}$ be a continuous function, Ω be an interval containing $(0, 1)$ and $\phi : \Omega \to \Omega$ be a function such that $\phi(x)(x + h) \in I$ for all $x \in \Omega$ and $h \in \mathbb{R}$. Then $(\phi - h)$-derivative is defined as follows:

$$ {}_hD_\phi(f(x)) = \frac{{}_hd_\phi f(x)}{{}_hd_\phi(x)} = \frac{f(\phi(x)(x+h)) - f(x)}{(\phi(x)-1)x + \phi(x)h}, x \neq \frac{\phi(x)h}{1-\phi(x)} := u, \tag{2.1} $$

where $\phi(x) \in (0, 1)$. If $x = u$, then we have ${}_hD_\phi(f(u)) = \lim_{x\to u}({}_hD_\phi(f(x)))$. When $h = 0$ in (2.1), then we have the concept of ϕ-derivative given as follows:

$$ D_\phi(f(x)) = \frac{f(x\phi(x)) - f(x)}{(\phi(x)-1)x}, x \neq 0, \text{ where } \phi(x) \in (0, 1). \tag{2.2} $$

When $\phi(x) = q$ and $h = 0$ in Definition 11, then we have q-derivative given in Definition 6, When $\phi(x) = q$ in he Definition 11, then we have $(q - h)$-derivative presented in Definition 7.

Example 1 Let $f(x) = x^n$, where n is positive integer. Take, $\phi(x) = x^2, \forall x \in (0, 1)$. Then

$$ \begin{aligned} {}_hD_\phi(x)^n &= \frac{(x^2(x+h))^n - x^n}{(x^2-1)x + x^2h} \\ &= \frac{((x^2)^n - 1)x^n}{(x^2-1)x + x^2h} + \frac{(x^2)^n\left[nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + h^n\right]}{(x^2-1)x + x^2h}. \end{aligned} \tag{2.3} $$

By taking $h = 0$, we have

$$ \begin{aligned} D_{x^2}(x)^n &= \frac{(x^2.x)^n - x^n}{(x^2-1)x} = \frac{((x^2)^n - 1)x^n}{(x^2-1)x} = x^{n-1}\frac{((x^2)^n - 1)}{(x^2-1)} \\ &= x^{n-1}((x^2)^{n-1} + (x^2)^{n-2} + \cdots + x^2 + 1). \end{aligned} \tag{2.4} $$

If $\phi(x) = q$, then we have $D_q(x)^n = [n]_q x^{n-1}$, $q \in (0, 1)$ where $[n]_q = \frac{1-q^n}{1-q}$.

Definition 12 Let $f : [a, b] \to \mathbb{R}$ be a continuous function, Ω an interval containing $(0, 1)$ and $\phi : \Omega \to \Omega$ be a function such that $\phi(x)(x + h) \in I, \forall x \in \Omega, h \in \mathbb{R}$, and $\phi(x) \in (0, 1)$. The left $(\phi - h)$-integral is defined as:

$$\int_a^x f(t)_h d_\phi t = ((1 - \phi(x))(x - a) + \phi(x)h) \sum_{n=0}^{\infty} \phi^n(x) f\left(\phi^n(x)a + (1 - \phi^n(x))x + nh\phi^n(x)\right), x > a, \quad (2.5)$$

and the right $(\phi - h)$-integral is defined as:

$$\int_x^b f(t)_h d_\phi t = ((1 - \phi(x))(b - x) + \phi(x)h) \sum_{n=0}^{\infty} \phi^n(x) f\left(\phi^n(x).x + (1 - \phi^n(x))b + +nh\phi^n(x)\right), b > x. \quad (2.6)$$

When $\phi(x) = q$ and $h = 0$ in Definition 12, then we have Definition 9, When $\phi(x) = q$ in Definition 12, then we have Definition 10.

Example 2 If $0 < \phi(x) < 1$, and $f(x) = x$ for all $x \in [a, b]$, then we have

$$\begin{aligned}
&\int_a^x f(x)_h d_\phi^a x \\
&= ((1 - \phi(x))(x - a) + \phi(x)h) \sum_{n=0}^{\infty} \phi^n(x) f(\phi^n(x).x + (1 - \phi^n(x))a + n\phi^n(x)h) \\
&= ((1 - \phi(x))(x - a) + \phi(x)h) \times \left(\sum_{n=0}^{\infty} \phi^{2n}(x).x + \sum_{n=0}^{\infty} (\phi^n(x) - \phi^{2n}(x))a + h \sum_{n=0}^{\infty} n\phi^{2n}(x)\right) \\
&= ((1 - \phi(x))(x - a) + \phi(x)h) \left(\frac{x + a\phi(x)}{1 - \phi^2(x)} + nh \frac{\phi^2(x)}{(1 - \phi^2(x))^2}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\int_x^b f(x)_h d_\phi^b x \\
&= ((1 - \phi(x))(b - x) + \phi(x)h) \sum_{n=0}^{\infty} \phi^n(x) f(\phi^n(x).x + (1 - \phi^n(x))b + n\phi^n(x)h) \\
&= ((1 - \phi(x))(b - x) + \phi(x)h) \\
&\quad \times \left(\sum_{n=0}^{\infty} \phi^{2n}(x).x + \sum_{n=0}^{\infty} (\phi^n(x) - \phi^{2n}(x))b + h \sum_{n=0}^{\infty} n\phi^{2n}(x)\right) \\
&= ((1 - \phi(x))(b - x) + \phi(x)h) \left(\frac{x + b\phi(x)}{1 - \phi^2(x)} + nh \frac{\phi^2(x)}{(1 - \phi^2(x))^2}\right).
\end{aligned}$$

If we take $f(x) = 1$ in Example 2, we get:

$$\int_a^x {}_h d_\phi^a x = \frac{(1-\phi(x))(x-a)+\phi(x)h}{1-\phi(x)}. \tag{2.7}$$

$$\int_x^b {}_h d_\phi^b x = \frac{(1-\phi(x))(b-x)+\phi(x)h}{1-\phi(x)}. \tag{2.8}$$

Example 3 Let $k \in \mathbb{N}$, $\phi(x) = x$ and $f(x) = x^k$, for all $x \in [a, b]$, then

$$\int_a^x x^k {}_h d_\phi x = ((1-\phi(x))(x-a)+\phi(x)h)\sum_{n=0}^{\infty}\phi^n(x)(\phi^n(x)(x+nh) + (1-\phi^n(x))a)^k$$

where

$$\begin{aligned}&(\phi^n(x)(x+nh)+(1-\phi^n(x))a)^k\\ &= \binom{k}{0}(\phi^n(x)(x+nh))^k((1-\phi^n(x))a)^0 + \binom{k}{1}(\phi^n(x)(x+nh))^{k-1}((1-\phi^n(x))a)^1\\ &\quad + \binom{k}{2}(\phi^n(x)(x+nh))^{k-2}((1-\phi^n(x))a)^2 + \cdots\\ &\quad + \binom{k}{k}(\phi^n(x)(x+nh))^0((1-\phi^n(x))a)^k.\end{aligned}$$

If we take $\phi(x) = x$, $x \in (0, 1)$ and $k = 3$, then we obtain that

$$\int_a^x x^3 {}_h d_x x = ((1-x)(x-a)+xh)\sum_{n=0}^{\infty} x^n(x^n(x+nh)+(1-x^n)a)^3$$

where

$$\begin{aligned}&(x^n(x+nh)+(1-x^n)a)^3\\ &= \binom{3}{0}(x^n(x+nh))^3((1-x^n)a)^0 + \binom{3}{1}(x^n(x+nh))^2((1-x^n)a)^1\\ &\quad + \binom{3}{2}(x^n(x+nh))^1((1-x^n)a)^2 + + \binom{3}{3}(x^n(x+nh))^0((1-x^n)a)^3.\end{aligned}$$

Hence, we have

$$\begin{aligned}\int_a^x x^3\,{}_h d_x x &= ((1-x)(x-a)+xh)[\sum_{n=0}^{\infty} x^n(x^n(x+nh))^3\\ &\quad+3\sum_{n=0}^{\infty} x^n(x^n(x+nh))^2((1-x^n)a)\\ &\quad+3\sum_{n=0}^{\infty} x^n(x^n(x+nh))((1-x^n)a)^2\\ &\quad+\sum_{n=0}^{\infty} x^n((1-x^n)a)^3].\end{aligned}$$

If $h=0$, then we get

$$\begin{aligned}\int_a^x x^3\,{}_0 d_x x &= (1-x)(x-a)\sum_{n=0}^{\infty} x^n(x^n.x+(1-x^n)a)^3\\ &= (1-x)(x-a)[x^3\sum_{n=0}^{\infty} x^{4n}+3x^2a\sum_{n=0}^{\infty}(x^{3n}-x^{4n})\\ &\quad+3xa^2\sum_{n=0}^{\infty} x^{2n}(1-x^n)^2+a^3\sum_{n=0}^{\infty} x^n(1-x^n)^3].\end{aligned}$$

As a special case, if we take $a=0$ and $\phi(x)=x=q$, then we have

$$\int_0^x x^3\,{}_0 d_q x = x^4\frac{1-q}{1-q^4}=\frac{x^4}{[4]_q}. \tag{2.9}$$

In the next section, we give new extensions of the q-Hermite–Hadamard inequality. Here, we will employ $(\phi-h)$-integrals to demonstrate inequalities for convex, m-convex, and $\hbar$-convex functions. Additionally, we will discuss theorems that have been proved using q-integrals.

3 Extended forms of the q-Hermite–Hadamard inequalities

Theorem 7 *Let $f:[a,b]\to\mathbb{R}$ be a convex function, Ω an interval containing $(0,1)$ and $\phi:\Omega\to\Omega$ a function such that $\phi(x)(x+h)\in[a,b]$, $\forall x\in\Omega$, and $h\in\mathbb{R}$. Then the following inequality holds for $(\phi-h)$-integrals:*

$$\begin{aligned}&f\left(\frac{a+\phi(x)x}{1+\phi(x)}+\frac{h\phi^2(x)}{(1-\phi^2(x))^2}(1-\phi(x))\right)\\ &\quad+f\left(\frac{x+\phi(x)b}{1+\phi(x)}+\frac{h\phi^2(x)}{(1-\phi^2(x))^2}(1-\phi(x))\right)\end{aligned}$$

$$\leq \frac{(1-\phi(x))}{(1-\phi(x))(x-a)+\phi(x)h}\int_a^x f(x)_h d_\phi x$$
$$+ \frac{(1-\phi(x))}{(1-\phi(x))(b-x)+\phi(x)h}\int_x^b f(x)_h d_\phi x$$
$$\leq \frac{(f(a)+\phi(x)f(b))(b-a)+(1+\phi(x))[(x-a)f(b)+(b-x)f(a)]}{(b-a)(1+\phi(x))}$$
$$+\left(\frac{f(b)-f(a)}{b-a}\right)\frac{2h\phi^2(x)}{(1-\phi^2(x))^2}(1-\phi(x)),\ where\, \phi(x)\in(0,1). \tag{3.1}$$

Proof Using the Jensen's inequality, we have

$$f\left(\sum_{n=0}^{\infty}(1-\phi(x))\phi^n(x)\left(\phi^n(x)a+(1-\phi^n(x))x+nh\phi^n(x)\right)\right)$$
$$\leq \sum_{n=0}^{\infty}(1-\phi(x))\phi^n(x)f\left(\phi^n(x)a+(1-\phi^n(x))x+nh\phi^n(x)\right), \tag{3.2}$$

where $\sum_{n=0}^{\infty}(1-\phi(x))\phi^n(x)=1$.
On simplifying the left hand side and applying the definition of left $(\phi-h)$-integral on right hand side of the inequality, we obtain that

$$f\left(\frac{a+\phi(x)x}{1+\phi(x)}+\frac{h\phi^2(x)}{(1-\phi^2(x))^2}(1-\phi(x))\right)$$
$$\leq \frac{(1-\phi(x))}{(1-\phi(x))(x-a)+h\phi(x)}\sum_{n=0}^{\infty}((1-\phi(x))(x-a)+h\phi(x))\,\phi^n(x)$$
$$\times f\left(\phi^n(x)a+(1-\phi^n(x))x+nh\phi^n(x)\right)$$
$$= \frac{(1-\phi(x))}{(1-\phi(x))(x-a)+h\phi(x)}\int_a^x f(x)_h d_\phi x. \tag{3.3}$$

Again, by using the Jensen's inequality, we get

$$f\left(\sum_{n=0}^{\infty}(1-\phi(x))\phi^n(x)\left(\phi^n(x).x+(1-\phi^n(x))b+nh\phi^n(x)\right)\right)$$
$$\leq \sum_{n=0}^{\infty}f\left((1-\phi(x))\phi^n(x)\left(\phi^n(x).x+(1-\phi^n(x))b+nh\phi^n(x)\right)\right). \tag{3.4}$$

On simplification the left side of the above inequality and using the definition of right $(\phi-h)$-integral on the right side, we get

$$
\begin{aligned}
&f\left(\frac{x+\phi(x)b}{1+\phi(x)}+\frac{h\phi^2(x)}{(1-\phi^2(x))^2}(1-\phi(x))\right)\\
&\leq \frac{(1-\phi(x))}{(1-\phi(x))(b-x)+h\phi(x)}\sum_{n=0}^{\infty}((1-\phi(x))(b-x)+h\phi(x))\,\phi^n(x)\\
&\quad\times f\left(\phi^n(x)x+(1-\phi^n(x))b+nh\phi^n(x)\right)\\
&=\frac{(1-\phi(x))}{(1-\phi(x))(b-x)+h\phi(x)}\int_x^b f(x)_h d_\phi x. \qquad (3.5)
\end{aligned}
$$

Note that, the left inequality of (3.1) is obtained by adding (3.3) and (3.5). Let $Z(x)$ be the chord joining two points $(a, f(a))$ and $(b, f(b))$. As f is a convex function, $f(x) \leq Z(x)$ which can be stated as:

$$
Z(x)=\frac{f(b)-f(a)}{b-a}x+\frac{bf(a)-af(b)}{b-a}. \qquad (3.6)
$$

That is,

$$
f(x)\leq \frac{bf(a)-af(b)}{b-a}+\frac{f(b)-f(a)}{b-a}x. \qquad (3.7)
$$

The left $(\phi-h)$ integration of (3.7) over the interval $[a, x)$ gives

$$
\begin{aligned}
\int_a^x f(x)_h d_\phi x &\leq \int_a^x \left(\frac{bf(a)-af(b)}{b-a}+\frac{f(b)-f(a)}{b-a}x\right)_h d_\phi x\\
&=((1-\phi(x))(x-a)+h\phi(x))\\
&\quad\times\left(\frac{(b-a)f(a)+\phi(x)f(a)(b-x)+\phi(x)f(b)(x-a)}{(b-a)(1-\phi^2(x))}\right.\\
&\quad\left.+\left(\frac{f(b)-f(a)}{b-a}\right)\frac{h\phi^2(x)}{(1-\phi^2(x))^2}\right), \qquad (3.8)
\end{aligned}
$$

which takes the following form:

$$
\begin{aligned}
\int_a^x f(x)_h d_\phi x &\leq ((1-\phi(x))(x-a)+h\phi(x))\\
&\quad\times\left(\frac{(b-a)f(a)+\phi(x)f(a)(b-x)+\phi(x)f(b)(x-a)}{(b-a)(1-\phi^2(x))}\right.\\
&\quad\left.+\left(\frac{f(b)-f(a)}{b-a}\right)\frac{h\phi^2(x)}{(1-\phi^2(x))^2}\right). \qquad (3.9)
\end{aligned}
$$

Moreover, the right $(\phi-h)$ integration of (3.7) over the interval $(x, b]$ gives

$$\begin{aligned}\int_x^b f(x)_h d_\phi x &\le \int_x^b \left(\frac{bf(a)-af(b)}{b-a}+\frac{f(b)-f(a)}{b-a}x\right)_h d_\phi x\\ &= ((1-\phi(x))(b-x)+h\phi(x))\\ &\quad\times\left(\frac{(b-a)\phi(x)f(b)+f(a)(b-x)+f(b)(x-a)}{(b-a)(1-\phi^2(x))}\right.\\ &\quad\left.+\left(\frac{f(b)-f(a)}{b-a}\right)\frac{h\phi^2(x)}{(1-\phi^2(x))^2}\right),\end{aligned} \tag{3.10}$$

which takes the following form:

$$\begin{aligned}\int_x^b f(x)_h d_\phi x &\le ((1-\phi(x))(b-x)+h\phi(x))\\ &\quad\times\left(\frac{(b-a)\phi(x)f(b)+f(a)(b-x)+f(b)(x-a)}{(b-a)(1-\phi^2(x))}\right.\\ &\quad\left.+\left(\frac{f(b)-f(a)}{b-a}\right)\frac{h\phi^2(x)}{(1-\phi^2(x))^2}\right).\end{aligned} \tag{3.11}$$

By combining (3.9) and (3.11), we obtain the second inequality of (3.1). □

Corollary 1 *Take $h=0$ in (3.1) and obtain the inequality for ϕ- integrals:*

$$\begin{aligned}&f\left(\frac{a+\phi(x)x}{1+\phi(x)}\right)+f\left(\frac{x+\phi(x)b}{1+\phi(x)}\right)\\ &\le\frac{1}{x-a}\int_a^x f(x)d_\phi x+\frac{1}{b-x}\int_x^b f(x)d_\phi x\\ &\le\frac{(f(a)+\phi(x)f(b))(b-a)+(1+\phi(x))[(x-a)f(b)+(b-x)f(a)]}{(b-a)(1+\phi(x))}.\end{aligned} \tag{3.12}$$

Remark 1 Setting $x=b$ in (3.3), $x=a$ in (3.5), and $x=b$ in (3.9), $x=a$, in (3.11), we get the following inequalities:

$$\begin{aligned}&f\left(\frac{a+\phi(x)b}{1+\phi(x)}+\frac{h\phi^2(x)}{(1-\phi^2(x))^2}(1-\phi(x))\right)\\ &\le\frac{1-\phi(x)}{(1-\phi(x))(b-a)+h\phi(x)}\int_a^b f(x)_h d_\phi x\\ &\le\frac{f(a)+\phi(x)f(b)}{1+\phi(x)}+\left(\frac{f(b)-f(a)}{b-a}\right)\frac{h\phi^2(x)}{(1-\phi^2(x))^2}(1-\phi(x)).\end{aligned} \tag{3.13}$$

Corollary 2 *Taking $h = 0$ in the Remark 1, the Hermite–Hadamard inequality for ϕ-integrals is given below:*

$$f\left(\frac{a+\phi(x)b}{1+\phi(x)}\right) \leq \frac{1}{b-a}\int_a^b f(x)d_\phi x \leq \frac{f(a)+\phi(x)f(b)}{1+\phi(x)}. \tag{3.14}$$

If we set, $\phi(x) = q$, we get q-Hermite–Hadamard inequality which reduces classical Hermite–Hadamard inequality on taking $q = 1$.

Using the Hermite–Hadamard inequality for $\hbar$-convex functions, we can now demonstrate the following generalization.

Theorem 8 *Let $f : [a, b] \to \mathbb{R}$ be $\hbar$-convex function such that $\hbar(\frac{1}{2}) \neq 0$, Ω an interval containing $(0, 1)$ and $\phi : \Omega \to \Omega$ be a function such that $\phi(x)(x+h) \in [a, b], \forall x \in \Omega$, and $h \in \mathbb{R}$. Then the following inequality holds for $(\phi - h)$-integral.*
(i) *The following inequality is applicable to the left $(\phi - h)$ -integral if f is symmetric about $\frac{a+x}{2}, \forall x \in (a, b)$.*

$$\begin{aligned}\frac{1}{2\hbar\left(\frac{1}{2}\right)} f\left(\frac{a+x}{2}\right) &\leq \frac{1-\phi(x)}{(1-\phi(x))(x-a)+\phi(x)\,h_1}\int_a^x f(\lambda)_{h_1}d_\phi\lambda \\ &\leq f(x)\int_0^1 \hbar(\lambda)\,{}_h d_\phi\lambda + f(a)\int_0^1 \hbar(1-\lambda)\,{}_h d_\phi\lambda, \\ &\text{where } h_1 = (x-a)h \text{ and } \phi(x) \in (0, 1).\end{aligned} \tag{3.15}$$

(ii) *The following inequality is applicable to the right $(\phi-h)$ -integral if f is symmetric about $\frac{x+b}{2}, \forall x \in (a, b)$.*

$$\begin{aligned}\frac{1}{2\hbar\left(\frac{1}{2}\right)} f\left(\frac{x+b}{2}\right) &\leq \frac{1-\phi(x)}{(1-\phi(x))(b-x)+\phi(x)\,h_2}\int_x^b f(\lambda)_{h_2}d_\phi\lambda \\ &\leq f(b)\int_0^1 \hbar(\lambda)\,{}_h d_\phi\lambda + f(x)\int_0^1 \hbar(1-\lambda)\,{}_h d_\phi\lambda, \\ &\text{where } h_2 = (b-x)h \text{ and } \phi(x) \in (0, 1).\end{aligned} \tag{3.16}$$

Proof We demonstrate (i) and (ii) in the following way: (i) Since f is $\hbar$-convex function, we get:

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} f\left(\frac{a+x}{2}\right) \leq f(\lambda a + (1-\lambda)x) + f(\lambda x + (1-\lambda)a), \forall \lambda \in [0, 1]. \tag{3.17}$$

Applying the $(\phi - h)$-integral on both sides over the interval $[0, 1]$:

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} f\left(\frac{a+x}{2}\right) \leq \frac{1-\phi(x)}{(1-\phi(x))+\phi(x)h}\left(\int_0^1 f(x-\lambda(x-a))_h d_\phi\lambda + \int_0^1 f(a+\lambda(x-a))_h d_\phi\lambda\right). \tag{3.18}$$

Since $f(a+x-u) = f(u), \forall u \in (a, x)$, we have $f(x-\lambda(x-a)) = f(a+\lambda(x-a))$. Then, the inequality (3.18) takes the following form:

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} f\left(\frac{a+x}{2}\right) \leq \frac{1-\phi(x)}{(1-\phi(x))+\phi(x)h}\int_0^1 f(a+\lambda(x-a))_h d_\phi\lambda. \tag{3.19}$$

Applying the left $(\phi - h)$-integral definition, we get:

$$\begin{aligned}
&\frac{(1-\phi(x))+\phi(x)h}{(1-\phi(x))(x-a)+\phi(x)\,h_1}\int_a^x f(\lambda)_{h_1} d_\phi\lambda \\
&= ((1-\phi(x))+\phi(x)h)\sum_{n=0}^{\infty}\phi^n(x) f\left(\phi^n(x)a + (1-\phi^n(x)).x + nh_1\phi^n(x)\right) \\
&= \int_0^1 f\left(a+(x-a)\lambda\right)_h d_\phi\lambda.
\end{aligned} \tag{3.20}$$

Now, using the $\hbar$-convexity of f on the right hand side of the equality (3.20), we have

$$\int_0^1 f\left(\lambda x + (1-\lambda)a\right)_h d_\phi\lambda \leq f(x)\int_0^1 \hbar(\lambda)_h d_\phi\lambda + f(a)\int_0^1 \hbar(1-\lambda)_h d_\phi\lambda.$$

Therefore, (3.20) becomes

$$\begin{aligned}
&\frac{(1-\phi(x))+\phi(x)h}{(1-\phi(x))(x-a)+\phi(x)\,h_1}\int_a^x f(\lambda)_{h_1} d_\phi\lambda \\
&\leq f(x)\int_0^1 \hbar(\lambda)_h d_\phi\lambda + f(a)\int_0^1 \hbar(1-\lambda)_h d_\phi\lambda.
\end{aligned} \tag{3.21}$$

Finally, we obtain the inequality (3.15) from (3.19), (3.20) and (3.21):
(ii) By using the $\hbar$-convexity, we get

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} f\left(\frac{x+b}{2}\right) \leq f(\lambda x + (1-\lambda)b) + f(\lambda b + (1-\lambda)x), \forall\lambda \in [0, 1]. \tag{3.22}$$

Applying the $(\phi - h)$-integral on the both sides over the interval $[0, 1]$, we have

$$\frac{1}{\hbar\left(\frac{1}{2}\right)} f\left(\frac{x+b}{2}\right) \leq \frac{1-\phi(x)}{(1-\phi(x))+\phi(x)h} \times \left(\int_0^1 f(b-\lambda(b-x))_h d_\phi\lambda + \int_0^1 f(x+\lambda(b-x))_h d_\phi\lambda\right). \tag{3.23}$$

Since $f(x+b-u) = f(u), \forall u \in (x, b)$, we have $f(b-\lambda(b-x)) = f(x+\lambda(b-x))$. The inequality (3.23) becomes:

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} f\left(\frac{x+b}{2}\right) \leq \frac{1-\phi(x)}{(1-\phi(x))+\phi(x)h} \int_0^1 f(x+\lambda(b-x))_h d_\phi\lambda. \tag{3.24}$$

We now apply the right $(\phi - h)$-integral to obtain the following

$$\begin{aligned}
&\frac{(1-\phi(x))+\phi(x)h}{(1-\phi(x))(b-x)+\phi(x)\,h_2} \int_x^b f(\lambda)_{h_2} d_\phi\lambda \\
&= ((1-\phi(x))+\phi(x)h) \sum_{n=0}^{\infty} \phi^n(x) f\left(\phi^n(x).x + (1-\phi^n(x))b + nh_2\phi^n(x)\right) \\
&= \int_0^1 f\,(x+\lambda(b-x))\;{}_h d_\phi\lambda.
\end{aligned} \tag{3.25}$$

As, f is $\hbar$-convex, the right hand side of equality (3.25) becomes

$$\int_0^1 f\,(x+\lambda(b-x))\;{}_h d_\phi\lambda \leq f(b)\int_0^1 \hbar(\lambda)\,{}_h d_\phi\lambda + f(x)\int_0^1 \hbar(1-\lambda)\,{}_h d_\phi\lambda.$$

So, (3.25) reduces to

$$\begin{aligned}
&\frac{(1-\phi(x))+\phi(x)h}{(1-\phi(x))(b-x)+\phi(x)\,h_2} \int_x^b f(\lambda)\,{}_{h_2} d_\phi\lambda \\
&\leq f(b)\int_0^1 \hbar(\lambda)\,{}_h d_\phi\lambda + f(x)\int_0^1 \hbar(1-\lambda)\,{}_h d_\phi\lambda.
\end{aligned} \tag{3.26}$$

We obtain the inequality (3.16) from (3.24), (3.25) and (3.26). □

Corollary 3 *Taking $h = 0$ in (3.15) and (3.16), we obtain the inequalities for the left and right ϕ-integrals as follows:*

$$\begin{aligned}
\frac{1}{2\hbar\left(\frac{1}{2}\right)} f\left(\frac{a+x}{2}\right) &\leq \frac{1}{x-a}\int_a^x f(\lambda) d_\phi\lambda \\
&\leq f(x)\int_0^1 \hbar(\lambda) d_\phi\lambda + f(a)\int_0^1 \hbar(1-\lambda) d_\phi\lambda
\end{aligned}$$

and

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)} f\left(\frac{x+b}{2}\right) \leq \frac{1}{b-x}\int_x^b f(\lambda)d_\phi\lambda \leq f(b)\int_0^1 \hbar(\lambda)d_\phi\lambda + f(x)\int_0^1 \hbar(1-\lambda)d_\phi\lambda.$$

Remark 2 By setting $x = b$ in (3.15) and $x = a$ in (3.16), we get the following inequality:

$$\begin{aligned}\frac{1}{2\hbar\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) &\leq \frac{1-\phi(x)}{(1-\phi(x))(b-a)+\phi(x)\,h_3}\int_a^b f(\lambda)_{h_3}d_\phi\lambda \\ &\leq f(b)\int_0^1 \hbar(\lambda)\,{}_h d_\phi\lambda + f(a)\int_0^1 \hbar(1-\lambda)\,{}_h d_\phi\lambda,\end{aligned} \tag{3.27}$$

where $h_3 = (b-a)h$.

Corollary 4 *Using $h = 0$ in the Remark 2, the inequality for ϕ -integrals is given as:*

$$\begin{aligned}\frac{1}{2\hbar\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b f(\lambda)d_\phi\lambda \\ &\leq f(b)\int_0^1 \hbar(\lambda)\,d_\phi\lambda + f(a)\int_0^1 \hbar(1-\lambda)d_\phi\lambda.\end{aligned} \tag{3.28}$$

Theorem 9 *Let $f : [0, y] \to \mathbb{R}$ be m-convex function, Ω an interval containing $(0, 1)$ and $\phi : \Omega \to \Omega$ be a function such that $\phi(x)(x + h) \in [0, y]$, $\forall x \in \Omega$, $h \in \mathbb{R}$ and $a, b \in [0, y]$ with $a < b$. Then,*
(i) *The following inequality holds for the left $(\phi - h)$ -integral provided that $f(\frac{a+x-u}{m}) = f(u), \forall u \in (a, x)$*

$$\begin{aligned}f\left(\frac{a+x}{2}\right) &\leq \frac{(1-\phi(x))(1+m)}{2((1-\phi(x))(x-a)+\phi(x)\,h_1)}\int_a^x f(\lambda)_{h_1}d_\phi\lambda \\ &\leq \frac{(1-\phi(x))+\phi(x)h}{2}\left(f(x)\left(\frac{\phi(x)}{1-\phi^2(x)}+\frac{\phi^2(x)h}{(1-\phi^2(x))^2}\right)\right. \\ &\quad \left. +mf\left(\frac{a}{m}\right)\left(\frac{1}{1-\phi^2(x)}+\frac{\phi^2(x)h}{(1-\phi^2(x))^2}\right)\right), \\ &\quad \text{where } h_1 = (x-a)h \text{ and } \phi(x) \in (0,1).\end{aligned} \tag{3.29}$$

(ii) *The following inequality holds for the right $(\phi - h)$ -integral if $f(\frac{x+b-u}{m}) = f(u), \forall u \in (x, b)$*

$$f\left(\frac{x+b}{2}\right) \leq \frac{(1-\phi(x))(1+m)}{2((1-\phi(x))(b-x)+\phi(x)\,h_2)} \int_x^b f(\lambda)_{h_2} d_\phi \lambda$$
$$\leq \frac{(1-\phi(x))+\phi(x)h}{2}\left(f(b)\left(\frac{\phi(x)}{1-\phi^2(x)}+\frac{\phi^2(x)h}{(1-\phi^2(x))^2}\right)\right.$$
$$\left.+mf\left(\frac{x}{m}\right)\left(\frac{1}{1-\phi^2(x)}-\frac{\phi^2(x)h}{(1-\phi^2(x))^2}\right)\right),$$
$$where\, h_2 = (b-x)h\, and\, \phi(x) \in (0,1). \tag{3.30}$$

Proof We now prove (i) and (ii) as follows:
(i) It follows from the m-convexity of f that

$$f\left(\frac{a+x}{2}\right) \leq \frac{1}{2} f(a+\lambda(x-a)) + \frac{m}{2} f\left(\frac{x-\lambda(x-a)}{m}\right), \forall \lambda \in [0,1]. \tag{3.31}$$

Using the fact: $f(\frac{x+a-u}{m}) = f(u), \forall u \in (a,x)$, we have $f\left(\frac{x-\lambda(x-a)}{m}\right) = f(a+\lambda(x-a))$. The above inequality becomes

$$f\left(\frac{a+x}{2}\right) \leq \frac{1+m}{2} f(a+\lambda(x-a)), \forall \lambda \in [0,1]. \tag{3.32}$$

Applying the left $(\phi-h)$-integral on the both sides over the interval $[0,1]$, we have

$$f\left(\frac{a+x}{2}\right) \leq \frac{(1-\phi(x))(1+m)}{2((1-\phi(x))+\phi(x)h)} \int_0^1 f(a+\lambda(x-a))_h d_\phi \lambda. \tag{3.33}$$

By using the $\hbar$-convexity of f, the right term of inequality (3.33) reduces to

$$\int_0^1 f(a+\lambda(x-a))\,{}_h d_\phi \lambda \leq f(x) \int_0^1 \lambda\,{}_h d_\phi \lambda + mf\left(\frac{a}{m}\right) \int_0^1 (1-\lambda)\,{}_h d_\phi \lambda \tag{3.34}$$

From (3.33), (3.20) and (3.34), we obtain the following:

$$f\left(\frac{a+x}{2}\right) \leq \frac{(1-\phi(x))(1+m)}{2((1-\phi(x))(x-a)+\phi(x)h_1)} \int_a^x f(\lambda)\,{}_{h_1} d_\phi \lambda$$
$$\leq f(x) \int_0^1 \lambda\,{}_h d_\phi \lambda + mf\left(\frac{a}{m}\right) \int_0^1 (1-\lambda)\,{}_h d_\phi \lambda. \tag{3.35}$$

On $(\phi-h)$-integrating (3.35) over the interval $[0,1]$, we get the inequality (3.29). Following arguments similar to those given above, we obtain (3.30) inequality from

(3.33) by using the definition of right $(\phi - h)$ -integral.

$$\begin{aligned} f\left(\frac{x+b}{2}\right) &\leq \frac{(1-\phi(x))(1+m)}{2((1-\phi(x))(b-x)+\phi(x)h_2)} \int_x^b f(\lambda)\,{}_{h_2}d_\phi\lambda \\ &\leq f(b)\int_0^1 \lambda\,{}_h d_\phi\lambda + mf\left(\frac{x}{m}\right)\int_0^1 (1-\lambda)\,{}_h d_\phi\lambda. \end{aligned} \tag{3.36}$$

$(\phi - h)$-integrating (3.36) over the interval [0, 1] gives the inequality (3.30). □

Corollary 5 *Taking $h = 0$ in (3.29) and (3.30), we obtain the inequalities for the left and right ϕ-integrals as follows:*

$$f\left(\frac{a+x}{2}\right) \leq \frac{(1+m)}{2(x-a)}\int_a^x f(\lambda)d_\phi\lambda \leq \frac{1}{2(1+\phi(x))}\left(\phi(x)f(x) + mf\left(\frac{a}{m}\right)\right), \tag{3.37}$$

$$f\left(\frac{b+x}{2}\right) \leq \frac{(1+m)}{2(b-x)}\int_x^b f(\lambda)d_\phi\lambda \leq \frac{1}{2(1+\phi(x))}\left(\phi(x)f(b) + mf\left(\frac{x}{m}\right)\right). \tag{3.38}$$

Remark 3 By setting $x = b$ in (3.29) and $x = a$ in (3.30), we get the following inequality:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{(1-\phi(x))(1+m)}{2((1-\phi(x))(b-a)+\phi(x)\,h_3)} \int_a^b f(\lambda)_{h_3} d_\phi\lambda \\ &\leq \frac{(1-\phi(x))+\phi(x)h}{2}\left(f(b)\left(\frac{\phi(x)}{1-\phi^2(x)} + \frac{\phi^2(x)h}{(1-\phi^2(x))^2}\right)\right. \\ &\quad \left. + mf\left(\frac{a}{m}\right)\left(\frac{1}{1-\phi^2(x)} + \frac{\phi^2(x)h}{(1-\phi^2(x))^2}\right)\right), \end{aligned} \tag{3.39}$$

where $h_3 = (b-a)h$.

Corollary 6 *If we take $h = 0$ in (3.40), then we have the inequality for ϕ -integral:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1+m}{2(b-a)}\int_a^b f(\lambda)d_\phi\lambda \leq \frac{1}{2(1+\phi(x))}\left(\phi(x)f(b) + mf\left(\frac{a}{m}\right)\right), \tag{3.40}$$

where $h_3 = (b-a)h$.

Remark 4 If we take $\phi(x) = q$; constant function, then the results in [7] are obtained.

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