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Fractional Euler-Maclaurin-type inequalities for various function classes

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Abstract
This paper investigates a technique that uses Riemann-Liouville fractional integrals to study several Euler-Maclaurin-type inequalities for various function classes. Afterwards, we provide our results by using special cases of obtained theorems and This paper is to derive examples. Moreover, we give some Euler-Maclaurin-type inequalities for bounded functions by fractional integrals. Furthermore, we construct some fractional Euler-Maclaurin-type inequalities for Lipschitzian functions. Finally, we offer some Euler-Maclaurin-type inequalities by fractional integrals of bounded variation.

Keywords Quadrature formulae · Maclaurin's formula · Convex functions · Fractional calculus

Mathematics Subject Classification 26D07 · 26D10 · 26D15 · 65D32

1 Introduction

In many branches of mathematics, inequality theory is well-known and continues to be a fascinating field of study with many applications. Furthermore, convex functions are important in the theory of inequality. Afterwards, because of its essential characteristics and practical applications, fractional calculus has drawn the attention of mathematicians in the mathematical sciences. Mathematicians have examined several fractional integral inequalities as a result of the significance of fractional calculus.

With applications in the theory of special means, Dragomir (1999) established an estimate of remainder for Simpson's quadratic formula in the case of bounded variation functions. Additionally, in paper Hezenci et al. (2021), several fractional Simpson-type inequalities were proved for functions whose second derivatives in absolute value are convex. Moreover, Budak et al. (2021a) considered several variations of Simpson-type inequalities using generalized fractional integrals in the context of differentiable convex functions. Readers can refer to

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Refs. (Budak et al. 2021b; Park 2013; Vivas-Cortez et al. 2020; Kashuri et al. 2020) and its references for additional information on Simpson-type inequalities and other features of Riemann–Liouville fractional integrals.

Since the three-point Newton-Cotes quadrature is a rule of Simpson's second rule, evaluations for three-step quadratic kernels are occasionally referred to as Newton type results in the literature. A large number of researchers have looked into Newton type inequalities thoroughly. For example, in paper Hezenci et al. (2023), various fractional Newton type inequalities for the case of bounded variation functions were presented and several Newton type inequalities for the case of differentiable convex functions were established using the Riemann-Liouville fractional integrals. Afterwards, Erden et al. (2020) established several novel Newton-type integral inequalities for functions whose first derivative is arithmetically-harmonically convex in absolute value at a given power. Likewise, Sitthiwirattham et al. (2022) provided some fractional Newton type inequalities for bounded variation functions. Additionally, Gao and Shi (2012) established certain applications for particular situations of real functions and indicated new Newton type inequality based on convexity. In order to obtain additional information on Newton-type inequalities, which include convex differentiable functions, please consult Refs. (Hezenci and Budak 2023; Iftikhar et al. 2020; Noor et al. 2018) and its references.

Using the Euler-Maclaurin-type inequalities, Lj et al. (2003) constructed a set of inequalities, and the results were applied to generate certain error estimates in the case of the Maclaurin quadrature rules. In addition, the Euler-Simpson 3/8 formulas were utilized to establish a set of inequalities. In paper Lj et al. (2011), the results were applied to provide some error estimates for the Simpson 3/8 quadrature rules. Moreover, in paper Hezenci and Budak, (2022), some Euler-Maclaurin-type inequalities were given in the case of differentiable convex functions. Afterwards, in paper Hezenci (2023), some corrected Euler-Maclaurin-type inequalities were proved using the Riemann-Liouville fractional integrals. For more details on these types of inequalities, the reader is referred to Refs. (Franjic et al. 2021; Davis and Rabinowitz 1975; Pečarić et al. 1992; Hezenci 2023) and the references therein.

The aim of this article is to derive Euler-Maclaurin-type inequalities for various function classes by using Riemann-Liouville fractional integrals. Section 2 provides a fundamental definition of fractional calculus as well as additional studies in this field. We will be proving an integral equality that is essential in order to establish the main findings of the article that is being given in Sect. 3. Using the Riemann-Liouville fractional integrals, various inequalities of the Euler-Maclaurin-type inequalities for differentiable convex functions will be presented in Sect. 4. Moreover, we will provide several graphical examples in order to demonstrate the accuracy of the newly established inequalities. In Sect. 5, we will give several Euler-Maclaurin-type for bounded functions by fractional integrals. In Sect. 6, some fractional Euler-Maclaurin-type will be established for Lipschitzian functions. In Sect. 7, Euler-Maclaurin-type will be proved by fractional integrals of bounded variation. In Sect. 8, we will discuss our opinions on Euler-Maclaurin-type inequalities and their potential consequences for future research areas.

2 Preliminaries

The *Riemann–Liouville integrals* $J^{\alpha}_{\sigma+}\mathcal{F}$ and $J^{\alpha}_{\delta-}\mathcal{F}$ of order $\alpha > 0$ with $\sigma \geq 0$ are defined by

$$J^{\alpha}_{\sigma+}\mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} \int_{\sigma}^{x} (x-\xi)^{\alpha-1} \mathcal{F}(\xi) d\xi, \quad x > \sigma$$

and

$$J^{\alpha}_{\delta-}\mathcal{F}(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\delta}(\xi - x)^{\alpha-1}\mathcal{F}(\xi)d\xi, \quad x < \delta,$$

respectively (Gorenflo and Mainardi 1997; Kilbas et al. 2006). Here, $\mathcal{F}$ belongs to $L_1[\sigma, \delta]$ and $\Gamma(\alpha)$ denotes the Gamma function defining as

$$\Gamma(\alpha) := \int_0^{\infty} e^{-u}u^{\alpha-1}du.$$

The fractional integral coincides with the classical integral for the case of $\alpha = 1$.

The following Simpson's rules apply to Simpson's inequalities:

1. The following is the formula for Simpson's quadrature, also known as Simpson's 1/3 rule:
$$\int_{\sigma}^{\delta}\mathcal{F}(x)\,dx \approx \frac{\delta-\sigma}{6}\left[\mathcal{F}(\sigma) + 4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \mathcal{F}(\delta)\right]. \tag{1}$$
2. The Newton-Cotes quadrature formula, commonly referred to as Simpson's second formula (also known as Simpson's 3/8 rule; see Davis and Rabinowitz (1975)), is expressed as follows:
$$\int_{\sigma}^{\delta}\mathcal{F}(x)\,dx \approx \frac{\delta-\sigma}{8}\left[\mathcal{F}(\sigma) + 3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right) + 3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right) + \mathcal{F}(\delta)\right]. \tag{2}$$
3. The Maclaurin rule, which is derived from the Maclaurin formula (see to Davis and Rabinowitz (1975)), is equivalent to the corresponding dual Simpson's 3/8 formula:
$$\int_{\sigma}^{\delta}\mathcal{F}(x)\,dx \approx \frac{\delta-\sigma}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right) + 2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + 3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]. \tag{3}$$

Formulae (1), (2), and (3) satisfy for all function $\mathcal{F}$ with continuous 4thderivative on $[\sigma, \delta]$.

The following is the most widely used Newton-Cotes quadrature that contains a three-point Simpson-type inequality:

Theorem 1 *Let $\mathcal{F} : [\sigma, \delta] \to \mathbb{R}$ be a four times differentiable and continuous function on (σ, δ), and let $\left\|\mathcal{F}^{(4)}\right\|_{\infty} = \sup_{x\in(\sigma,\delta)}\left|\mathcal{F}^{(4)}(x)\right| < \infty$. Then, the following inequality holds:*

$$\left|\frac{1}{6}\left[\mathcal{F}(\sigma) + 4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \mathcal{F}(\delta)\right] - \frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(x)dx\right| \le \frac{1}{2880}\left\|\mathcal{F}^{(4)}\right\|_{\infty}(\delta-\sigma)^4.$$

Based on the Simpson 3/8 inequality, the Simpson 3/8 rule is a classical closed type quadrature rule that looks like this:

Theorem 2 *Let us consider that $\mathcal{F} : [\sigma, \delta] \to \mathbb{R}$ is a four times differentiable and continuous function on (σ, δ), and $\left\|\mathcal{F}^{(4)}\right\|_{\infty} = \sup_{x\in(\sigma,\delta)}\left|\mathcal{F}^{(4)}(x)\right| < \infty$. Then, one has the inequality*

$$\left|\frac{1}{8}\left[\mathcal{F}(\sigma) + 3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right) + 3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right) + \mathcal{F}(\delta)\right] - \frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(x)dx\right|$$
$$\le \frac{1}{6480}\left\|\mathcal{F}^{(4)}\right\|_{\infty}(\delta-\sigma)^4.$$

The Maclaurin rule, which is derived from the Maclaurin inequality, is equivalent to the corresponding dual Simpson's 3/8 formula:

Theorem 3 *Let $\mathcal{F} : [\sigma, \delta] \to \mathbb{R}$ denote a four times differentiable and continuous function on (σ, δ), and let $\left\|\mathcal{F}^{(4)}\right\|_\infty = \sup_{x\in(\sigma,\delta)} \left|\mathcal{F}^{(4)}(x)\right| < \infty$. Then, the following inequality holds:*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{1}{\delta-\sigma}\int_\sigma^\delta \mathcal{F}(x)dx\right| \leq \frac{7}{51840}\left\|\mathcal{F}^{(4)}\right\|_\infty (\delta-\sigma)^4.$$

3 Principal outcome

In this section, we prove integral equality in order to illustrate the main results of this study.

Lemma 1 *Note that $\mathcal{F} : [\sigma, \delta] \to \mathbb{R}$ is an absolutely continuous function (σ, δ) such that $\mathcal{F}' \in L_1[\sigma, \delta]$. Then, the following equality holds:*

$$\begin{aligned}&\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]\\&\quad-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^\alpha}\left[J^\alpha_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J^\alpha_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\\&=\frac{\delta-\sigma}{4}\left[I_1+I_2\right].\end{aligned}$$

Here,

$$\begin{cases} I_1 = \int_0^{\frac{2}{3}} \left(\xi^\alpha - \frac{1}{4}\right)\left[\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)-\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right]d\xi,\\ I_2 = \int_{\frac{2}{3}}^1 \left(\xi^\alpha - 1\right)\left[\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)-\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right]d\xi. \end{cases}$$

Proof Using the integration by parts, we can quickly arrive

$$\begin{aligned}I_1 &= \int_0^{\frac{2}{3}} \left(\xi^\alpha-\frac{1}{4}\right)\left[\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)-\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right]d\xi\\&=\frac{2}{\delta-\sigma}\left(\xi^\alpha-\frac{1}{4}\right)\left[\mathcal{F}\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)+\mathcal{F}\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right]\Big|_0^{\frac{2}{3}}\\&\quad-\frac{2\alpha}{\delta-\sigma}\int_0^{\frac{2}{3}}\xi^{\alpha-1}\left[\mathcal{F}\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)+\mathcal{F}\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right]d\xi\\&=\frac{2}{\delta-\sigma}\left(\left(\frac{2}{3}\right)^\alpha-\frac{1}{4}\right)\left[\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]+\frac{1}{\delta-\sigma}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\\&\quad-\frac{2\alpha}{\delta-\sigma}\int_0^{\frac{2}{3}}\xi^{\alpha-1}\left[\mathcal{F}\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)+\mathcal{F}\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right]d\xi. \quad (4)\end{aligned}$$

According to the information given, it follows that

$$I_2 = \int_{\frac{2}{3}}^{1} \left(\xi^{\alpha} - 1\right) \left[\mathcal{F}'\left(\frac{1+\xi}{2}\delta + \frac{1-\xi}{2}\sigma\right) - \mathcal{F}'\left(\frac{1+\xi}{2}\sigma + \frac{1-\xi}{2}\delta\right)\right] d\xi$$
$$= -\frac{2}{\delta-\sigma}\left(\left(\frac{2}{3}\right)^{\alpha} - 1\right)\left[\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right) + \mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]$$
$$- \frac{2\alpha}{\delta-\sigma}\int_{\frac{2}{3}}^{1} \xi^{\alpha-1}\left[\mathcal{F}\left(\frac{1+\xi}{2}\delta + \frac{1-\xi}{2}\sigma\right) + \mathcal{F}\left(\frac{1+\xi}{2}\sigma + \frac{1-\xi}{2}\delta\right)\right] d\xi. \quad (5)$$

If we combine (4) and (5), then we readily have

$$I_1 + I_2 = \frac{1}{2(\delta-\sigma)}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right) + 2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + 3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]$$
$$- \frac{2\alpha}{\delta-\sigma}\int_{0}^{1} \xi^{\alpha-1}\left[\mathcal{F}\left(\frac{1+\xi}{2}\delta + \frac{1-\xi}{2}\sigma\right) + \mathcal{F}\left(\frac{1+\xi}{2}\sigma + \frac{1-\xi}{2}\delta\right)\right]. \quad (6)$$

If we use the change of the variable $x = \frac{1+\xi}{2}\delta + \frac{1-\xi}{2}\sigma$ and $y = \frac{1+\xi}{2}\sigma + \frac{1-\xi}{2}\delta$ for $\xi \in [0,1]$, then the equality (6) can be rewritten as follows

$$I_1 + I_2 = \frac{1}{2(\delta-\sigma)}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right) + 2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + 3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]$$
$$- \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha+1}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]. \quad (7)$$

Consequently, multiplying both sides of (7) by $\frac{\delta-\sigma}{4}$ concludes the proof of Lemma 1. □

4 Convex functions: fractional Euler-Maclaurin-type inequalities

Theorem 4 *Considering that Lemma 1 holds and the function $|\mathcal{F}'|$ is convex on the interval $[\sigma, \delta]$. Then, one can establish fractional Euler-Maclaurin-type inequality*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right) + 2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + 3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]\right.$$
$$\left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|$$
$$\leq \frac{\delta-\sigma}{4}\left(\Omega_1(\alpha) + \Omega_2(\alpha)\right)\left[\left|\mathcal{F}'(\sigma)\right| + \left|\mathcal{F}'(\delta)\right|\right]. \quad (8)$$

Here,

$$\Omega_1(\alpha) = \int_{0}^{\frac{2}{3}} \left|\xi^{\alpha} - \frac{1}{4}\right| d\xi = \begin{cases} \frac{2\alpha}{\alpha+1}\left(\frac{1}{4}\right)^{1+\frac{1}{\alpha}} + \frac{1}{\alpha+1}\left(\frac{2}{3}\right)^{\alpha+1} - \frac{1}{6}, & 0 < \alpha < \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}, \\ \frac{1}{6} - \frac{1}{\alpha+1}\left(\frac{2}{3}\right)^{\alpha+1}, & \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)} < \alpha, \end{cases}$$

$$\Omega_2(\alpha) = \int_{\frac{2}{3}}^{1} \left(1-\xi^{\alpha}\right) d\xi = \frac{1}{3} - \frac{1}{\alpha+1} + \frac{1}{\alpha+1}\left(\frac{2}{3}\right)^{\alpha+1}.$$

Proof If we consider the absolute value in Lemma 1, we can directly get

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]\right.\\
&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\
&\leq \frac{\delta-\sigma}{4}\left\{\int_{0}^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|\left|\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)-\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right|d\xi\right.\\
&\quad\left.+\int_{\frac{2}{3}}^{1}\left|\xi^{\alpha}-1\right|\left|\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)-\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right|d\xi\right\}. \qquad (9)
\end{aligned}$$

Since $\left|\mathcal{F}'\right|$ is convex, we have

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]\right.\\
&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\
&\leq \frac{\delta-\sigma}{4}\left\{\int_{0}^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|\left[\left(\frac{1+\xi}{2}\right)\left|\mathcal{F}'(\delta)\right|+\left(\frac{1-\xi}{2}\right)\left|\mathcal{F}'(\sigma)\right|+\left(\frac{1+\xi}{2}\right)\left|\mathcal{F}'(\sigma)\right|\right.\right.\\
&\quad\left.+\left(\frac{1-\xi}{2}\right)\left|\mathcal{F}'(\delta)\right|\right]d\xi\\
&\quad+\int_{\frac{2}{3}}^{1}\left(1-\xi^{\alpha}\right)\left[\left(\frac{1+\xi}{2}\right)\left|\mathcal{F}'(\delta)\right|+\left(\frac{1-\xi}{2}\right)\left|\mathcal{F}'(\sigma)\right|+\left(\frac{1+\xi}{2}\right)\left|\mathcal{F}'(\sigma)\right|\right.\\
&\quad\left.\left.+\left(\frac{1-\xi}{2}\right)\left|\mathcal{F}'(\delta)\right|\right]d\xi\right\}\\
&=\frac{\delta-\sigma}{4}\left(\Omega_1(\alpha)+\Omega_2(\alpha)\right)\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right].
\end{aligned}$$

□

Remark 1 If we choose $\alpha = 1$ in Theorem 4, then we can obtain Euler-Maclaurin-type inequality

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(\xi)\,d\xi\right|\\
&\quad\leq\frac{25(\delta-\sigma)}{576}\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right],
\end{aligned}$$

which is established in paper (Hezenci and Budak, 2022, Corollary 1).

Example 1 If a function $\mathcal{F} : [\sigma, \delta] = [0, 2] \to \mathbb{R}$ is defined by $\mathcal{F}(x) = x^2$ with $\alpha \in (0, 20]$, then the left-hand side of (8) coincides with

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]\right.\\
&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\
&=\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{1}{3}\right)+2\mathcal{F}(1)+3\mathcal{F}\left(\frac{5}{3}\right)\right]-\frac{\Gamma(\alpha+1)}{2}\left[J^{\alpha}_{0+}\mathcal{F}(1)+J^{\alpha}_{2-}\mathcal{F}(1)\right]\right|\\
&=\left|\frac{4}{3}-\frac{\alpha}{2}\left[\int_0^1(1-\xi)^{\alpha-1}\xi^2 d\xi+\int_1^2(\xi-1)^{\alpha-1}\xi^2 d\xi\right]\right|=\left|\frac{2(1-\alpha)}{3(\alpha+2)}\right|. \qquad (10)
\end{aligned}$$

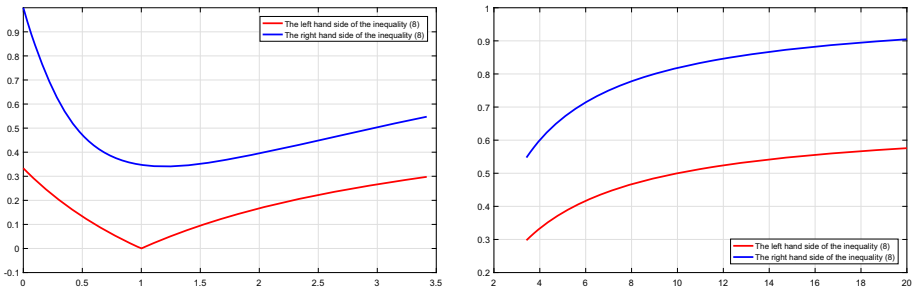
The right hand-side of (8) reduces to

$$\begin{aligned}
&\frac{\delta-\sigma}{4}\left(\Omega_1(\alpha)+\Omega_2(\alpha)\right)\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right]=2\left(\Omega_1(\alpha)+\Omega_2(\alpha)\right)\\
&=\begin{cases}\frac{4\alpha}{(\alpha+1)}\left(\frac{1}{4}\right)^{1+\frac{1}{\alpha}}+\frac{4}{\alpha+1}\left(\frac{2}{3}\right)^{\alpha+1}-\frac{2}{\alpha+1}+\frac{1}{3}, & 0<\alpha<\frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)},\\ \frac{\alpha-1}{\alpha+1}, & \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}<\alpha\leq 20.\end{cases}
\end{aligned}$$

Consequently, we have

$$\begin{cases}\left|\frac{2(1-\alpha)}{3(\alpha+2)}\right|\leq\frac{4\alpha}{(\alpha+1)}\left(\frac{1}{4}\right)^{1+\frac{1}{\alpha}}+\frac{4}{\alpha+1}\left(\frac{2}{3}\right)^{\alpha+1}-\frac{2}{\alpha+1}+\frac{1}{3}, & 0<\alpha<\frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)},\\ \left|\frac{2(1-\alpha)}{3(\alpha+2)}\right|\leq\frac{\alpha-1}{\alpha+1}, & \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}<\alpha\leq 20.\end{cases}$$

As one can see in Fig. 1, the left-hand side of the given inequalities from above is always below the right-hand side of the given inequalities.



(A) Graph on the interval $0 < \alpha \leq \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}$. (B) Graph on the interval $\frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)} < \alpha \leq 20$.

Fig. 1 The left-hand side of (8) in Example 1 is consistently below the right-hand side of (8), as seen in **A** and **B**, for all values of $\alpha \in (0, 20]$

Theorem 5 *Let's examine the assumptions in Lemma 1 and the function* $\left|\mathcal{F}'\right|^q$, $q > 1$ *is convex on* $[\sigma, \delta]$. *Then, the following Euler-Maclaurin-type inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]\right.\\
&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\
&\leq\frac{\delta-\sigma}{4}\left\{\left(\int_0^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|^p d\xi\right)^{\frac{1}{p}}\right.\\
&\quad\left[\left(\frac{4\left|\mathcal{F}'(\delta)\right|^q+2\left|\mathcal{F}'(\sigma)\right|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{4\left|\mathcal{F}'(\sigma)\right|^q+2\left|\mathcal{F}'(\delta)\right|^q}{9}\right)^{\frac{1}{q}}\right]\\
&\quad+\left(\int_{\frac{2}{3}}^1\left(1-\xi^{\alpha}\right)^p d\xi\right)^{\frac{1}{p}}\left[\left(\frac{11\left|\mathcal{F}'(\delta)\right|^q+\left|\mathcal{F}'(\sigma)\right|^q}{36}\right)^{\frac{1}{q}}\right.\\
&\quad\left.\left.+\left(\frac{11\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\delta)\right|^q}{36}\right)^{\frac{1}{q}}\right]\right\}. \qquad (11)
\end{aligned}$$

Here, $\frac{1}{p}+\frac{1}{q}=1$.

Proof By applying Hölder's inequality to (9), we have

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right.\right.\\
&\quad\left.\left.+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\
&\leq\frac{\delta-\sigma}{4}\left\{\left(\int_0^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^{\frac{2}{3}}\left|\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)\right|^q d\xi\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_0^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^{\frac{2}{3}}\left|\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right|^q d\xi\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{2}{3}}^1\left|\xi^{\alpha}-1\right|^p d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^1\left|\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)\right|^q d\xi\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\int_{\frac{2}{3}}^{1}\left|\xi^{\alpha}-1\right|^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\left|\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right|^{q}d\xi\right)^{\frac{1}{q}}\Bigg\}.$$

Taking advantage of the convexity $\left|\mathcal{F}'\right|^{q}$, we can easily find

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]\right.\\
&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\
&\leq\frac{\delta-\sigma}{4}\left\{\left(\int_{0}^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{2}{3}}\left(\frac{1+\xi}{2}\left|\mathcal{F}'(\delta)\right|^{q}+\frac{1-\xi}{2}\left|\mathcal{F}'(\sigma)\right|^{q}\right)d\xi\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_{0}^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{2}{3}}\left(\frac{1+\xi}{2}\left|\mathcal{F}'(\sigma)\right|^{q}+\frac{1-\xi}{2}\left|\mathcal{F}'(\delta)\right|^{q}\right)d\xi\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{2}{3}}^{1}\left(1-\xi^{\alpha}\right)^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\left(\frac{1+\xi}{2}\left|\mathcal{F}'(\delta)\right|^{q}+\frac{1-\xi}{2}\left|\mathcal{F}'(\sigma)\right|^{q}\right)d\xi\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{2}{3}}^{1}\left(1-\xi^{\alpha}\right)^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\left(\frac{1+\xi}{2}\left|\mathcal{F}'(\sigma)\right|^{q}+\frac{1-\xi}{2}\left|\mathcal{F}'(\delta)\right|^{q}\right)d\xi\right)^{\frac{1}{q}}\right\}\\
&=\frac{\delta-\sigma}{4}\left\{\left(\int_{0}^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|^{p}d\xi\right)^{\frac{1}{p}}\left[\left(\frac{4\left|\mathcal{F}'(\delta)\right|^{q}+2\left|\mathcal{F}'(\sigma)\right|^{q}}{9}\right)^{\frac{1}{q}}\right.\right.\\
&\quad\left.+\left(\frac{4\left|\mathcal{F}'(\sigma)\right|^{q}+2\left|\mathcal{F}'(\delta)\right|^{q}}{9}\right)^{\frac{1}{q}}\right]\\
&\quad+\left(\int_{\frac{2}{3}}^{1}\left(1-\xi^{\alpha}\right)^{p}d\xi\right)^{\frac{1}{p}}\left[\left(\frac{11\left|\mathcal{F}'(\delta)\right|^{q}+\left|\mathcal{F}'(\sigma)\right|^{q}}{36}\right)^{\frac{1}{q}}\right.\\
&\quad\left.\left.+\left(\frac{11\left|\mathcal{F}'(\sigma)\right|^{q}+\left|\mathcal{F}'(\delta)\right|^{q}}{36}\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}$$

□

Corollary 1 *Note that $\alpha = 1$ in Theorem 5. Then, we obtain*

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(\xi)\,d\xi\right| \\
&\leq \frac{\delta-\sigma}{4}\left\{\left(\frac{1}{p+1}\left[\left(\frac{1}{4}\right)^{p+1}+\left(\frac{5}{12}\right)^{p+1}\right]\right)^{\frac{1}{p}}\right. \\
&\quad\times\left[\left(\frac{4\left|\mathcal{F}'(\delta)\right|^{q}+2\left|\mathcal{F}'(\sigma)\right|^{q}}{9}\right)^{\frac{1}{q}}+\left(\frac{4\left|\mathcal{F}'(\sigma)\right|^{q}+2\left|\mathcal{F}'(\delta)\right|^{q}}{9}\right)^{\frac{1}{q}}\right] \\
&\quad+\left(\frac{1}{p+1}\left(\frac{1}{3}\right)^{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{11\left|\mathcal{F}'(\delta)\right|^{q}+\left|\mathcal{F}'(\sigma)\right|^{q}}{36}\right)^{\frac{1}{q}}\right. \\
&\quad\left.\left.+\left(\frac{11\left|\mathcal{F}'(\sigma)\right|^{q}+\left|\mathcal{F}'(\delta)\right|^{q}}{36}\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}$$

Example 2 Let us consider a function $\mathcal{F}:[\sigma,\delta]=[0,2]\to\mathbb{R}$ given by $\mathcal{F}(x)=x^2$. From Theorem 5 with $\alpha\in(0,20]$ and $p=q=2$, the left-hand side of (11) becomes to equality (10) and the right hand-side of (11) equals to

$$\begin{aligned}
&\left[\frac{1}{2\alpha+1}\left(\frac{2}{3}\right)^{2\alpha+1}-\frac{1}{2(\alpha+1)}\left(\frac{2}{3}\right)^{\alpha+1}+\frac{1}{24}\right]^{\frac{1}{2}}\left(\frac{4+2\sqrt{2}}{3}\right) \\
&\quad+\left[\frac{1}{3}+\frac{1}{2\alpha+1}\left(1-\left(\frac{2}{3}\right)^{2\alpha+1}\right)-\frac{2}{\alpha+1}\left(1-\left(\frac{2}{3}\right)^{\alpha+1}\right)\right]^{\frac{1}{2}}\left(\frac{1+\sqrt{11}}{3}\right).
\end{aligned}$$

Finally, we get the inequality

$$\begin{aligned}
\left|\frac{2(1-\alpha)}{3(\alpha+2)}\right| &\leq \left[\frac{1}{2\alpha+1}\left(\frac{2}{3}\right)^{2\alpha+1}-\frac{1}{2(\alpha+1)}\left(\frac{2}{3}\right)^{\alpha+1}+\frac{1}{24}\right]^{\frac{1}{2}}\left(\frac{4+2\sqrt{2}}{3}\right) \\
&\quad+\left[\frac{1}{3}+\frac{1}{2\alpha+1}\left(1-\left(\frac{2}{3}\right)^{2\alpha+1}\right)-\frac{2}{\alpha+1}\left(1-\left(\frac{2}{3}\right)^{\alpha+1}\right)\right]^{\frac{1}{2}}\left(\frac{1+\sqrt{11}}{3}\right).
\end{aligned}$$

It is easy to confirm that the left-hand side of (11) in Example 2 is always lower than the right-hand side of (11) in Fig. 2 for all values of $\alpha\in(0,20]$ using MATLAB software Figs. 3,4.

Theorem 6 *Suppose that the assumptions of Lemma 1 hold and the function $\left|\mathcal{F}'\right|^{q}$, $q\geq 1$ is convex on $[\sigma,\delta]$. Then, we have the following Euler-Maclaurin-type inequality*

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]\right. \\
&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|
\end{aligned}$$

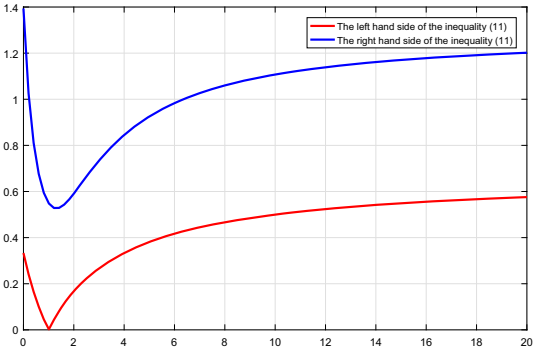


Fig. 2 MATLAB has been applied to compute and plot the graph of both sides of (11) in Example 2, depending on α

$$\leq \frac{\delta-\sigma}{4}\left\{(\Omega_1(\alpha))^{1-\frac{1}{q}}\left[\left[\Omega_3(\alpha)\left|\mathcal{F}'(\delta)\right|^q+\Omega_4(\alpha)\left|\mathcal{F}'(\sigma)\right|^q\right]^{\frac{1}{q}}\right.\right.$$
$$\left.+\left[\Omega_3(\alpha)\left|\mathcal{F}'(\sigma)\right|^q+\Omega_4(\alpha)\left|\mathcal{F}'(\delta)\right|^q\right]^{\frac{1}{q}}\right]$$
$$+(\Omega_2(\alpha))^{1-\frac{1}{q}}\left[\left[\Omega_5(\alpha)\left|\mathcal{F}'(\delta)\right|^q+\Omega_6(\alpha)\left|\mathcal{F}'(\sigma)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.\left.+\left[\Omega_5(\alpha)\left|\mathcal{F}'(\sigma)\right|^q+\Omega_6(\alpha)\left|\mathcal{F}'(\delta)\right|^q\right]^{\frac{1}{q}}\right]\right\}. \tag{12}$$

Here, $\Omega_1(\alpha)$ and $\Omega_2(\alpha)$ are specified in Theorem 4 and

$$\Omega_3(\alpha)=\int_0^{\frac{2}{3}}\frac{1+\xi}{2}\left|\xi^\alpha-\frac{1}{4}\right|d\xi$$
$$=\begin{cases}\frac{\alpha}{\alpha+1}\left(\frac{1}{4}\right)^{1+\frac{1}{\alpha}}+\frac{1}{2(\alpha+1)}\left(\frac{2}{3}\right)^{\alpha+1}\\ +\frac{\alpha+1}{2(\alpha+2)}\left(\frac{1}{4}\right)^{1+\frac{2}{\alpha}}+\frac{1}{2(\alpha+2)}\left(\frac{2}{3}\right)^{\alpha+2}-\frac{1}{9}, & 0<\alpha<\frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)},\\ \frac{1}{9}-\frac{1}{2(\alpha+1)}\left(\frac{2}{3}\right)^{\alpha+1}-\frac{1}{2(\alpha+2)}\left(\frac{2}{3}\right)^{\alpha+2}, & \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}<\alpha,\end{cases}$$

$$\Omega_4(\alpha)=\int_0^{\frac{2}{3}}\frac{1-\xi}{2}\left|\xi^\alpha-\frac{1}{4}\right|d\xi$$
$$=\begin{cases}\frac{\alpha}{\alpha+1}\left(\frac{1}{4}\right)^{1+\frac{1}{\alpha}}+\frac{1}{2(\alpha+1)}\left(\frac{2}{3}\right)^{\alpha+1}\\ -\frac{\alpha+1}{2(\alpha+2)}\left(\frac{1}{4}\right)^{1+\frac{2}{\alpha}}-\frac{1}{2(\alpha+2)}\left(\frac{2}{3}\right)^{\alpha+2}-\frac{1}{18}, & 0<\alpha<\frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)},\\ \frac{1}{18}-\frac{1}{2(\alpha+1)}\left(\frac{2}{3}\right)^{\alpha+1}+\frac{1}{2(\alpha+2)}\left(\frac{2}{3}\right)^{\alpha+2}, & \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}<\alpha,\end{cases}$$

$$\Omega_5(\alpha)=\int_{\frac{2}{3}}^1\left(\frac{1+\xi}{2}\right)\left(1-\xi^\alpha\right)d\xi=\frac{11}{36}-\frac{2\alpha+3}{2(\alpha+1)(\alpha+2)}+\frac{1}{2(\alpha+1)}\left(\frac{2}{3}\right)^{\alpha+1}$$

$$+\frac{1}{2(\alpha+2)}\left(\frac{2}{3}\right)^{\alpha+2},$$

$$\Omega_6(\alpha) = \int_{\frac{2}{3}}^{1}\left(\frac{1-\xi}{2}\right)\left(1-\xi^{\alpha}\right)d\xi = \frac{1}{36}-\frac{1}{2(\alpha+1)(\alpha+2)}+\frac{1}{2(\alpha+1)}\left(\frac{2}{3}\right)^{\alpha+1}$$
$$-\frac{1}{2(\alpha+2)}\left(\frac{2}{3}\right)^{\alpha+2}.$$

Proof When we first apply (9) to the power-mean inequality, we obtain

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]\right.\\
&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\
&\leq\frac{\delta-\sigma}{4}\left\{\left(\int_0^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|d\xi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|\left|\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)\right|^{q}d\xi\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_0^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|d\xi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|\left|\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right|^{q}d\xi\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{2}{3}}^{1}\left|\xi^{\alpha}-1\right|d\xi\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left|\xi^{\alpha}-1\right|\left|\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)\right|^{q}d\xi\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{2}{3}}^{1}\left|\xi^{\alpha}-1\right|d\xi\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left|\xi^{\alpha}-1\right|\left|\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right|^{q}d\xi\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Utilizing the convexity of $\left|\mathcal{F}'\right|^{q}$, it generates

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]\right.\\
&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\
&\leq\frac{\delta-\sigma}{4}\left\{\left(\int_0^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|d\xi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|\left[\left(\frac{1+\xi}{2}\right)\left|\mathcal{F}'(\delta)\right|^{q}\right.\right.\right.\\
&\quad\left.\left.\left.+\left(\frac{1-\xi}{2}\right)\left|\mathcal{F}'(\sigma)\right|^{q}\right]d\xi\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_0^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|d\xi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|\left[\left(\frac{1+\xi}{2}\right)\left|\mathcal{F}'(\sigma)\right|^{q}+\left(\frac{1-\xi}{2}\right)\left|\mathcal{F}'(\delta)\right|^{q}\right]d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^{1}\left(1-\xi^{\alpha}\right)d\xi\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left|\xi^{\alpha}-1\right|\left[\left(\frac{1+\xi}{2}\right)\left|\mathcal{F}'(\delta)\right|^{q}+\left(\frac{1-\xi}{2}\right)\left|\mathcal{F}'(\sigma)\right|^{q}\right]d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^{1}\left(1-\xi^{\alpha}\right)d\xi\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left|\xi^{\alpha}-1\right|\left[\left(\frac{1+\xi}{2}\right)\left|\mathcal{F}'(\sigma)\right|^{q}+\left(\frac{1-\xi}{2}\right)\left|\mathcal{F}'(\delta)\right|^{q}\right]d\xi\right)^{\frac{1}{q}}\Bigg\}$$

$$=\frac{\delta-\sigma}{4}\Bigg\{\left(\Omega_1(\alpha)\right)^{1-\frac{1}{q}}\left[\left(\left[\Omega_3(\alpha)\left|\mathcal{F}'(\delta)\right|^{q}+\Omega_4(\alpha)\left|\mathcal{F}'(\sigma)\right|^{q}\right]\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left[\Omega_3(\alpha)\left|\mathcal{F}'(\sigma)\right|^{q}+\Omega_4(\alpha)\left|\mathcal{F}'(\delta)\right|^{q}\right]\right)^{\frac{1}{q}}\right]$$

$$+\left(\Omega_2(\alpha)\right)^{1-\frac{1}{q}}\left[\left(\left[\Omega_5(\alpha)\left|\mathcal{F}'(\delta)\right|^{q}+\Omega_6(\alpha)\left|\mathcal{F}'(\sigma)\right|^{q}\right]\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left[\Omega_5(\alpha)\left|\mathcal{F}'(\sigma)\right|^{q}+\Omega_6(\alpha)\left|\mathcal{F}'(\delta)\right|^{q}\right]\right)^{\frac{1}{q}}\right]\Bigg\}.$$

□

Remark 2 If we assign $\alpha = 1$ in Theorem 6, then we have the following Euler-Maclaurin-type inequality

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(\xi)\,d\xi\right|$$

$$\leq\frac{\delta-\sigma}{72}\Bigg\{\left(\frac{17}{8}\right)^{1-\frac{1}{q}}\left[\left(\frac{863\left|\mathcal{F}'(\delta)\right|^{q}+361\left|\mathcal{F}'(\sigma)\right|^{q}}{576}\right)^{\frac{1}{q}}\right.$$

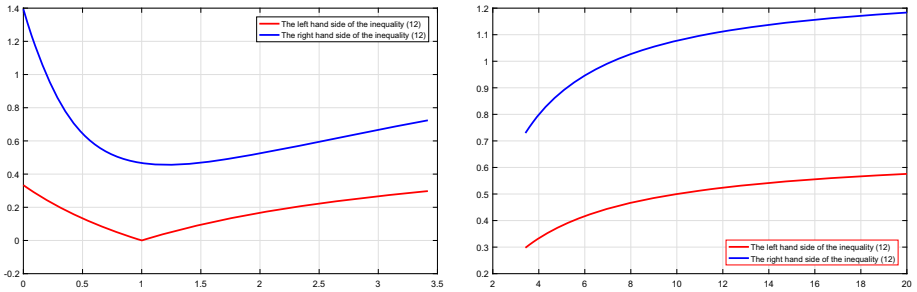
$$\left.+\left(\frac{863\left|\mathcal{F}'(\sigma)\right|^{q}+361\left|\mathcal{F}'(\delta)\right|^{q}}{576}\right)^{\frac{1}{q}}\right]$$

$$+\left[\left(\frac{8\left|\mathcal{F}'(\delta)\right|^{q}+\left|\mathcal{F}'(\sigma)\right|^{q}}{9}\right)^{\frac{1}{q}}+\left(\frac{8\left|\mathcal{F}'(\sigma)\right|^{q}+\left|\mathcal{F}'(\delta)\right|^{q}}{9}\right)^{\frac{1}{q}}\right]\Bigg\},$$

which is proved in paper (Hezenci and Budak, 2022, Corollary 3).

Example 3 If a function $\mathcal{F}:[\sigma,\delta]=[0,2]\to\mathbb{R}$ is presented by $\mathcal{F}(x)=x^2$. From Theorem 6 with $\alpha\in(0,20]$ and $q=2$, the left-hand side of (12) coincides with equality (10) and the right hand-side of (12) becomes to

$$2\left\{\left(\Omega_1(\alpha)\right)^{\frac{1}{2}}\left[\left[\Omega_3(\alpha)\right]^{\frac{1}{2}}+\left[\Omega_4(\alpha)\right]^{\frac{1}{2}}\right]+\left(\Omega_2(\alpha)\right)^{\frac{1}{2}}\left[\left[\Omega_5(\alpha)\right]^{\frac{1}{2}}+\left[\Omega_6(\alpha)\right]^{\frac{1}{2}}\right]\right\}. \quad (13)$$



(A) Graph on the interval $0 < \alpha \leq \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}$. (B) Graph on the interval $\frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)} < \alpha \leq 20$.

Fig. 3 In Example 3, using MATLAB software, **A** and **B** make it clear that the left-hand side of (12) constantly stays below the right-hand side

5 Bounded functions: Euler-Maclaurin-type inequalities with fractional integrals

In this section, we deal with some Euler-Maclaurin-type inequalities for bounded functions via fractional integrals.

Theorem 7 *Note that the conditions of Lemma 1 hold. If there exist* $m, M \in \mathbb{R}$ *such that* $m \leq \mathcal{F}'(\xi) \leq M$ *for* $\xi \in [\sigma, \delta]$, *then it follows*

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]\right. \\
&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right| \\
&\leq \frac{\delta-\sigma}{4}\left\{\Omega_1(\alpha)+\Omega_2(\alpha)\right\}(M-m). \qquad (14)
\end{aligned}$$

Proof By using the Lemma 1, we have

$$\begin{aligned}
&\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right] \\
&\quad-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right] \\
&=\frac{\delta-\sigma}{4}\left\{\int_0^{\frac{2}{3}}\left(\xi^{\alpha}-\frac{1}{4}\right)\left[\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)-\frac{m+M}{2}\right]d\xi\right. \\
&\quad+\int_0^{\frac{2}{3}}\left(\xi^{\alpha}-\frac{1}{4}\right)\left[\frac{m+M}{2}-\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right]d\xi \\
&\quad\left.+\int_{\frac{2}{3}}^{1}\left(\xi^{\alpha}-1\right)\left[\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)-\frac{m+M}{2}\right]d\xi\right.
\end{aligned}$$

$$+\int_{\frac{2}{3}}^{1}\left(\xi^{\alpha}-1\right)\left[\frac{m+M}{2}-\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right]d\xi\Bigg\}. \tag{15}$$

Through the absolute value of (15), we get

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]\right.\\
&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J_{\delta-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\
&\leq\frac{\delta-\sigma}{4}\left\{\int_{0}^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|\left|\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)-\frac{m+M}{2}\right|d\xi\right.\\
&\quad+\int_{0}^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|\left|\frac{m+M}{2}-\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right|d\xi\\
&\quad+\int_{\frac{2}{3}}^{1}\left(1-\xi^{\alpha}\right)\left|\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)-\frac{m+M}{2}\right|d\xi\\
&\quad\left.+\int_{\frac{2}{3}}^{1}\left(1-\xi^{\alpha}\right)\left|\frac{m+M}{2}-\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right|d\xi\right\}.
\end{aligned}$$

It is known that $m\leq\mathcal{F}'(\xi)\leq M$ for $\xi\in[\sigma,\delta]$. Then, we have

$$\left|\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)-\frac{m+M}{2}\right|\leq\frac{M-m}{2} \tag{16}$$

and

$$\left|\frac{m+M}{2}-\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right|\leq\frac{M-m}{2}. \tag{17}$$

With the help of the (16) and (17), we obtain

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]\right.\\
&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J_{\delta-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|\\
&\leq\frac{\delta-\sigma}{4}(M-m)\left\{\int_{0}^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|d\xi+\int_{\frac{2}{3}}^{1}\left(1-\xi^{\alpha}\right)d\xi\right\}\\
&=\frac{\delta-\sigma}{4}\left\{\Omega_1(\alpha)+\Omega_2(\alpha)\right\}(M-m).
\end{aligned}$$

□

Corollary 2 *If we select* $\alpha = 1$ *in Theorem 7, then we get*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right] -\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(\xi)d\xi\right| \leq \frac{25\,(\delta-\sigma)}{576}\,(M-m)\,.$$

Corollary 3 *Under assumption of Theorem 7, if there exist* $M \in \mathbb{R}^{+}$ *such that* $\left|\mathcal{F}'(\xi)\right| \leq M$ *for all* $\xi \in [\sigma, \delta]$ *, then we have*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right] -\frac{2^{\alpha-1}\Gamma\,(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right| \leq \frac{\delta-\sigma}{2}\left\{\Omega_1\,(\alpha)+\Omega_2\,(\alpha)\right\}M.$$

Corollary 4 *Let us consider* $\alpha = 1$ *in Corollary 3. Then, the following inequality holds:*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right] -\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(\xi)d\xi\right| \leq \frac{25\,(\delta-\sigma)}{288}M.$$

Example 4 Note that a function $\mathcal{F} : [\sigma, \delta] = [0, 2] \to \mathbb{R}$ is presented by $\mathcal{F}(x) = x^2$. From Theorem 7 with $\alpha \in (0, 20]$ and $0 \leq \mathcal{F}'(\xi) \leq 4$, the left-hand side of (14) coincides with equality (10) and the right hand-side of (14) is

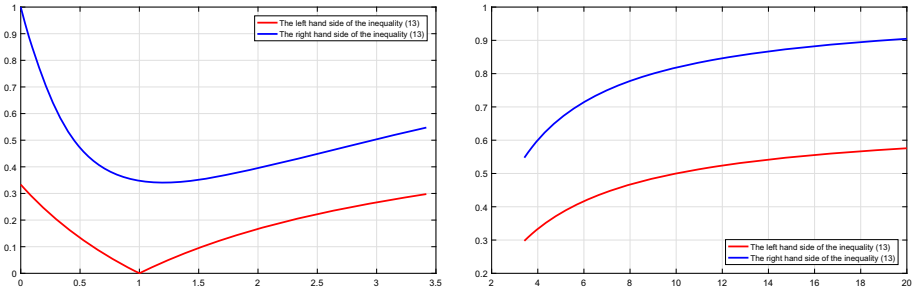
$$\begin{cases} \frac{4\alpha}{(\alpha+1)}\left(\frac{1}{4}\right)^{1+\frac{1}{\alpha}}+\frac{4}{\alpha+1}\left(\frac{2}{3}\right)^{\alpha+1}-\frac{2}{\alpha+1}+\frac{1}{3}, & 0<\alpha<\frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}, \\ \frac{\alpha-1}{\alpha+1}, & \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}<\alpha\leq 20. \end{cases}$$

Finally, we get the inequalities

$$\begin{cases} \left|\frac{2(1-\alpha)}{3(\alpha+2)}\right| \leq \frac{4\alpha}{(\alpha+1)}\left(\frac{1}{4}\right)^{1+\frac{1}{\alpha}}+\frac{4}{\alpha+1}\left(\frac{2}{3}\right)^{\alpha+1}-\frac{2}{\alpha+1}+\frac{1}{3}, & 0<\alpha<\frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}, \\ \left|\frac{2(1-\alpha)}{3(\alpha+2)}\right| \leq \frac{\alpha-1}{\alpha+1}, & \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}<\alpha\leq 20. \end{cases}$$

6 Lipschitzian functions: fractional Euler-Maclaurin-type inequalities

In this section, we give some fractional Euler-Maclaurin-type inequalities for Lipschitzian functions.



(A) Graph on the interval $0 < \alpha \leq \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}$. (B) Graph on the interval $\frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)} < \alpha \leq 20$.

Fig. 4 **A** and **B** make it clear that, for all values of $\alpha \in (0, 20]$, the left-hand side of equality (14) in Example 4 constantly stays below the right-hand side

Theorem 8 *Assume that the assumptions of Lemma 1 are valid. If $\mathcal{F}'$ is a L-Lipschitzian function on $[\sigma, \delta]$, then the following inequality holds:*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|$$
$$\leq \frac{(\delta-\sigma)^2}{4}L\left\{\Omega_7(\alpha)+\Omega_8(\alpha)\right\}.$$

Here,

$$\Omega_7(\alpha)=\int_0^{\frac{2}{3}}\xi\left|\xi^{\alpha}-\frac{1}{4}\right|d\xi=\begin{cases}\frac{\alpha+1}{\alpha+2}\left(\frac{1}{4}\right)^{1+\frac{2}{\alpha}}+\frac{1}{\alpha+2}\left(\frac{2}{3}\right)^{\alpha+2}-\frac{1}{18}, & 0<\alpha<\frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)},\\ \frac{1}{18}-\frac{1}{\alpha+2}\left(\frac{2}{3}\right)^{\alpha+2}, & \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}<\alpha\end{cases}$$

and

$$\Omega_8(\alpha)=\int_{\frac{2}{3}}^{1}\xi\left|\xi^{\alpha}-1\right|d\xi=\frac{5}{18}-\frac{1}{\alpha+2}+\frac{1}{\alpha+2}\left(\frac{2}{3}\right)^{\alpha+2}.$$

Proof With the aid of Lemma 1 and since $\mathcal{F}'$ is L -Lipschitzian function, we have

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|$$
$$=\left|\frac{\delta-\sigma}{4}\left\{\int_0^{\frac{2}{3}}\left(\xi^{\alpha}-\frac{1}{4}\right)\left[\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)-\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right]d\xi\right.\right.$$

$$+\int_{\frac{2}{3}}^{1}\left(\xi^{\alpha}-1\right)\left[\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)-\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right]d\xi\Bigg\}\Bigg|$$

$$\leq\frac{\delta-\sigma}{4}\left\{\int_{0}^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|\left|\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)-\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right|d\xi\right.$$

$$\left.+\int_{\frac{2}{3}}^{1}\left(1-\xi^{\alpha}\right)\left|\mathcal{F}'\left(\frac{1+\xi}{2}\delta+\frac{1-\xi}{2}\sigma\right)-\mathcal{F}'\left(\frac{1+\xi}{2}\sigma+\frac{1-\xi}{2}\delta\right)\right|d\xi\right\}\Bigg|$$

$$\leq\frac{\delta-\sigma}{4}\left\{\int_{0}^{\frac{2}{3}}\left|\xi^{\alpha}-\frac{1}{4}\right|L\xi\,(\delta-\sigma)\,d\xi+\int_{\frac{2}{3}}^{1}\left(1-\xi^{\alpha}\right)L\xi\,(\delta-\sigma)\,d\xi\right\}$$

$$=\frac{(\delta-\sigma)^{2}}{4}L\left\{\Omega_{7}(\alpha)+\Omega_{8}(\alpha)\right\}.$$

□

Corollary 5 *Consider $\alpha=1$ in Theorem 8. Then, the following Euler-Maclaurin-type inequality holds:*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(\xi)d\xi\right|\leq\frac{475\,(\delta-\sigma)^{2}}{20736}L.$$

7 Functions of bounded variation: Euler-Maclaurin-type inequalities via fractional integrals

In this section, we represent Euler-Maclaurin-type inequalities by fractional integrals of bounded variation.

Theorem 9 *Consider that $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ is a function of bounded variation on $[\sigma,\delta]$. Then, we have*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J_{\delta-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right|$$

$$\leq\frac{1}{2}\max\left\{1-\left(\frac{2}{3}\right)^{\alpha},\left|\frac{1}{4}-\left(\frac{1}{3}\right)^{\alpha}\right|,\frac{1}{4}\right\}\bigvee_{\sigma}^{\delta}(\mathcal{F}).$$

Here, $\bigvee_{c}^{d}(\mathcal{F})$ denotes the total variation of $\mathcal{F}$ on $[c,d]$.

Proof Define the function $K_\alpha(x)$ by

$$K_\alpha(x) = \begin{cases} \left(\frac{\delta-\sigma}{2}\right)^\alpha - \left(\frac{\sigma+\delta}{2} - x\right)^\alpha, & \sigma \le x < \frac{5\sigma+\delta}{6}, \\ \frac{1}{4}\left(\frac{\delta-\sigma}{2}\right)^\alpha - \left(\frac{\sigma+\delta}{2} - x\right)^\alpha, & \frac{5\sigma+\delta}{6} \le x < \frac{\sigma+\delta}{2}, \\ \left(x - \frac{\sigma+\delta}{2}\right)^\alpha - \frac{1}{4}\left(\frac{\delta-\sigma}{2}\right)^\alpha, & \frac{\sigma+\delta}{2} \le x < \frac{\sigma+5\delta}{6}, \\ \left(x - \frac{\sigma+\delta}{2}\right)^\alpha - \left(\frac{\delta-\sigma}{2}\right)^\alpha, & \frac{\sigma+5\delta}{6} \le x \le \delta. \end{cases}$$

With the aid of the integrating by parts, we have

$$\begin{aligned} \int_\sigma^\delta K_\alpha(x)d\mathcal{F}(x) &= \int_\sigma^{\frac{5\sigma+\delta}{6}} \left[\left(\frac{\delta-\sigma}{2}\right)^\alpha - \left(\frac{\sigma+\delta}{2} - x\right)^\alpha\right] d\mathcal{F}(x) \\ &+ \int_{\frac{5\sigma+\delta}{6}}^{\frac{\sigma+\delta}{2}} \left[\frac{1}{4}\left(\frac{\delta-\sigma}{2}\right)^\alpha - \left(\frac{\sigma+\delta}{2} - x\right)^\alpha\right] d\mathcal{F}(x) \\ &+ \int_{\frac{\sigma+\delta}{2}}^{\frac{\sigma+5\delta}{6}} \left[\left(x - \frac{\sigma+\delta}{2}\right)^\alpha - \frac{1}{4}\left(\frac{\delta-\sigma}{2}\right)^\alpha\right] d\mathcal{F}(x) \\ &+ \int_{\frac{\sigma+5\delta}{6}}^{\delta} \left[\left(x - \frac{\sigma+\delta}{2}\right)^\alpha - \left(\frac{\delta-\sigma}{2}\right)^\alpha\right] d\mathcal{F}(x) \\ &= \left[\left(\frac{\delta-\sigma}{2}\right)^\alpha - \left(\frac{\sigma+\delta}{2} - x\right)^\alpha\right] \mathcal{F}(x)\Big|_\sigma^{\frac{5\sigma+\delta}{6}} - \alpha \int_\sigma^{\frac{5\sigma+\delta}{6}} \left(\frac{\sigma+\delta}{2} - x\right)^{\alpha-1} \mathcal{F}(x)dx \\ &+ \left[\frac{1}{4}\left(\frac{\delta-\sigma}{2}\right)^\alpha - \left(\frac{\sigma+\delta}{2} - x\right)^\alpha\right] \mathcal{F}(x)\Big|_{\frac{5\sigma+\delta}{6}}^{\frac{\sigma+\delta}{2}} - \alpha \int_{\frac{5\sigma+\delta}{6}}^{\frac{\sigma+\delta}{2}} \left(\frac{\sigma+\delta}{2} - x\right)^{\alpha-1} \mathcal{F}(x)dx \\ &+ \left[\left(x - \frac{\sigma+\delta}{2}\right)^\alpha - \frac{1}{4}\left(\frac{\delta-\sigma}{2}\right)^\alpha\right] \mathcal{F}(x)\Big|_{\frac{\sigma+\delta}{2}}^{\frac{\sigma+5\delta}{6}} - \alpha \int_{\frac{\sigma+\delta}{2}}^{\frac{\sigma+5\delta}{6}} \left(x - \frac{\sigma+\delta}{2}\right)^{\alpha-1} \mathcal{F}(x)dx \\ &+ \left[\left(x - \frac{\sigma+\delta}{2}\right)^\alpha - \left(\frac{\delta-\sigma}{2}\right)^\alpha\right] \mathcal{F}(x)\Big|_{\frac{\sigma+5\delta}{6}}^{\delta} - \alpha \int_{\frac{\sigma+5\delta}{6}}^{\delta} \left(x - \frac{\sigma+\delta}{2}\right)^{\alpha-1} \mathcal{F}(x)dx \\ &= \frac{3}{4}\left(\frac{\delta-\sigma}{2}\right)^\alpha \mathcal{F}\left(\frac{5\sigma+\delta}{6}\right) + \frac{3}{4}\left(\frac{\delta-\sigma}{2}\right)^\alpha \mathcal{F}\left(\frac{\sigma+5\delta}{6}\right) + \frac{1}{2}\left(\frac{\delta-\sigma}{2}\right)^\alpha \mathcal{F}\left(\frac{\sigma+\delta}{2}\right) \end{aligned}$$

$$
\begin{aligned}
&- \alpha \int_{\sigma}^{\frac{\sigma+\delta}{2}} \left(\frac{\sigma+\delta}{2} - x\right)^{\alpha-1} \mathcal{F}(x)dx - \alpha \int_{\frac{\sigma+\delta}{2}}^{\delta} \left(x - \frac{\sigma+\delta}{2}\right)^{\alpha-1} \mathcal{F}(x)dx \\
&= \frac{(\delta-\sigma)^{\alpha}}{2^{\alpha-1}} \frac{1}{8} \left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right) + 2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + 3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right] \\
&\quad - \Gamma(\alpha+1)\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]. \qquad (18)
\end{aligned}
$$

In other words, we have

$$
\begin{aligned}
&\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right) + 2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + 3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right] \\
&\quad - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right] \\
&\quad = \frac{2^{\alpha-1}}{(\delta-\sigma)^{\alpha}} \int_{\sigma}^{\delta} K_{\alpha}(x)d\mathcal{F}(x).
\end{aligned}
$$

It is known that if $\mathcal{G}, \mathcal{F} : [\sigma, \delta] \to \mathbb{R}$ are such that $\mathcal{G}$ is continuous on $[\sigma, \delta]$ and $\mathcal{F}$ is of bounded variation on $[\sigma, \delta]$, then $\int_{\sigma}^{\delta} \mathcal{G}(\xi)d\mathcal{F}(\xi)$ exists and

$$
\left|\int_{\sigma}^{\delta} \mathcal{G}(\xi)d\mathcal{F}(\xi)\right| \leq \sup_{\xi\in[\sigma,\delta]} |\mathcal{G}(\xi)| \bigvee_{\sigma}^{\delta}(\mathcal{F}). \qquad (19)
$$

By using (19), it yields

$$
\begin{aligned}
&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right) + 2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + 3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]\right. \\
&\quad \left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + J^{\alpha}_{\delta-}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]\right| \\
&\quad = \frac{2^{\alpha-1}}{(\delta-\sigma)^{\alpha}}\left|\int_{\sigma}^{\delta} K_{\alpha}(x)d\mathcal{F}(x)\right| \\
&\quad \leq \frac{2^{\alpha-1}}{(\delta-\sigma)^{\alpha}}\left\{\left|\int_{\sigma}^{\frac{5\sigma+\delta}{6}} \left[\left(\frac{\delta-\sigma}{2}\right)^{\alpha} - \left(\frac{\sigma+\delta}{2} - x\right)^{\alpha}\right]d\mathcal{F}(x)\right|\right. \\
&\quad\quad + \left|\int_{\frac{5\sigma+\delta}{6}}^{\frac{\sigma+\delta}{2}} \left[\frac{1}{4}\left(\frac{\delta-\sigma}{2}\right)^{\alpha} - \left(\frac{\sigma+\delta}{2} - x\right)^{\alpha}\right]d\mathcal{F}(x)\right| \\
&\quad\quad \left. + \left|\int_{\frac{\sigma+\delta}{2}}^{\frac{\sigma+5\delta}{6}} \left[\left(x - \frac{\sigma+\delta}{2}\right)^{\alpha} - \frac{1}{4}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}\right]d\mathcal{F}(x)\right|\right.
\end{aligned}
$$

$$+\left|\int_{\frac{\sigma+5\delta}{6}}^{\delta}\left[\left(x-\frac{\sigma+\delta}{2}\right)^{\alpha}-\left(\frac{\delta-\sigma}{2}\right)^{\alpha}\right]d\mathcal{F}(x)\right|\Bigg\}$$

$$\leq \frac{2^{\alpha-1}}{(\delta-\sigma)^{\alpha}}\Bigg\{\sup_{x\in\left[\sigma,\frac{5\sigma+\delta}{6}\right]}\left|\left(\frac{\delta-\sigma}{2}\right)^{\alpha}-\left(\frac{\sigma+\delta}{2}-x\right)^{\alpha}\right|\bigvee_{\sigma}^{\frac{5\sigma+\delta}{6}}(\mathcal{F})$$

$$+\sup_{x\in\left[\frac{5\sigma+\delta}{6},\frac{\sigma+\delta}{2}\right]}\left|\frac{1}{4}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}-\left(\frac{\sigma+\delta}{2}-x\right)^{\alpha}\right|\bigvee_{\frac{5\sigma+\delta}{6}}^{\frac{\sigma+\delta}{2}}(\mathcal{F})$$

$$+\sup_{x\in\left[\frac{\sigma+\delta}{2},\frac{\sigma+5\delta}{6}\right]}\left|\left(x-\frac{\sigma+\delta}{2}\right)^{\alpha}-\frac{1}{4}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}\right|\bigvee_{\frac{\sigma+\delta}{2}}^{\frac{\sigma+5\delta}{6}}(\mathcal{F})$$

$$+\sup_{x\in\left[\frac{\sigma+5\delta}{6},\delta\right]}\left|\left(x-\frac{\sigma+\delta}{2}\right)^{\alpha}-\left(\frac{\delta-\sigma}{2}\right)^{\alpha}\right|\bigvee_{\frac{\sigma+5\delta}{6}}^{\delta}(\mathcal{F})\Bigg\}$$

$$=\frac{2^{\alpha-1}}{(\delta-\sigma)^{\alpha}}\Bigg\{\left[\left(\frac{\delta-\sigma}{2}\right)^{\alpha}-\left(\frac{\delta-\sigma}{3}\right)^{\alpha}\right]\bigvee_{\sigma}^{\frac{5\sigma+\delta}{6}}(\mathcal{F})$$

$$+\max\left[\left|\frac{1}{4}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}-\left(\frac{\delta-\sigma}{3}\right)^{\alpha}\right|,\frac{1}{4}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}\right]\bigvee_{\frac{5\sigma+\delta}{6}}^{\frac{\sigma+\delta}{2}}(\mathcal{F})$$

$$+\max\left[\left|\frac{1}{4}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}-\left(\frac{\delta-\sigma}{3}\right)^{\alpha}\right|,\frac{1}{4}\left(\frac{\delta-\sigma}{2}\right)^{\alpha}\right]\bigvee_{\frac{\sigma+\delta}{2}}^{\frac{\sigma+5\delta}{6}}(\mathcal{F})$$

$$+\left[\left(\frac{\delta-\sigma}{2}\right)^{\alpha}-\left(\frac{\delta-\sigma}{3}\right)^{\alpha}\right]\bigvee_{\frac{\sigma+5\delta}{6}}^{\delta}(\mathcal{F})\Bigg\}$$

$$=\frac{1}{2}\Bigg\{\left[1-\left(\frac{2}{3}\right)^{\alpha}\right]\bigvee_{\sigma}^{\frac{5\sigma+\delta}{6}}(\mathcal{F})+\max\left[\left|\frac{1}{4}-\left(\frac{2}{3}\right)^{\alpha}\right|,\frac{1}{4}\right]\bigvee_{\frac{5\sigma+\delta}{6}}^{\frac{\sigma+\delta}{2}}(\mathcal{F})$$

$$+\max\left[\left|\frac{1}{4}-\left(\frac{2}{3}\right)^{\alpha}\right|,\frac{1}{4}\right]\bigvee_{\frac{\sigma+\delta}{2}}^{\frac{\sigma+5\delta}{6}}(\mathcal{F})+\left[1-\left(\frac{2}{3}\right)^{\alpha}\right]\bigvee_{\frac{\sigma+5\delta}{6}}^{\delta}(\mathcal{F})\Bigg\}$$

$$\leq\frac{1}{2}\max\left\{1-\left(\frac{2}{3}\right)^{\alpha},\left|\frac{1}{4}-\left(\frac{2}{3}\right)^{\alpha}\right|,\frac{1}{4}\right\}\bigvee_{\sigma}^{\delta}(\mathcal{F}).$$

□

Corollary 6 *Let us consider* $\alpha=1$ *in Theorem* 9*. Then, the following inequality holds:*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(\xi)d\xi\right|\leq\frac{5}{24}\bigvee_{\sigma}^{\delta}(\mathcal{F}).$$

8 Conclusion

This paper is to derive Euler-Maclaurin-type inequalities for various function classes by using Riemann-Liouville fractional integrals. First of all, we give an integral equality that is essential in order to establish the main findings of the article. Using the Riemann-Liouville fractional integrals, some Euler-Maclaurin-type inequalities are investigated for differentiable convex functions. Moreover, we present several examples using graphs in order to show that our main result is correct. In addition to this, we give some Euler-Maclaurin-type for bounded functions by fractional integrals. Furthermore, some fractional Euler-Maclaurin-type inequalities are considered for Lipschitzian functions. Finally, Euler-Maclaurin-type inequalities are proved by fractional integrals of bounded variation.

In future studies, improvements or generalizations of our results can be investigated by using different kinds of convex function classes or other types of fractional integral operators.

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Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

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